

# HOMOTOPY POISSON-N ALGEBRAS FROM N-PLECTIC STRUCTURES

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**ABSTRACT.** We associate a homotopy Poisson- $n$  algebra to any higher symplectic structure, which generalizes the common symplectic Poisson algebra of smooth functions. This provides robust  $n$ -plectic prequantum data for most approaches to quantization.

## 1. INTRODUCTION

The basic notion of higher symplectic geometry is very simple: Instead of symplectic 2-forms, consider closed forms of arbitrary degree.

This idea, however, has a remarkable long history. It only gradually emerged over the last few decades, mostly from an attempt to find a covariant Hamilton formalism for physical fields. In fact first origins can be traced back to the work of Vito Volterra [19], which was published as early as 1890. Many different flavors appeared ever since, most of them with a strong motivation from real world applications.

Despite all the effort, one has to admit, that many such theories have a certain anachronistic touch. Often no full analog of the symplectic Poisson algebra is known and this renders most of them very different from their symplectic counterparts. In fact it effectively prohibits allot symplectic techniques, let alone the known approaches to quantization.

The present work aims to fill that gap. We show that the natural structure to expect from higher symplectic data is not a Poisson, but a *homotopy Poisson- $n$*  algebra.

Homotopy Poisson- $n$  algebras are relatively modern structures and an explicit description was only found recently [5]. Nevertheless they are very pleasant in the sense, that they provide robust prequantum data. The abstract theory says, that they are the homology of certain  $\mathbb{E}_n$ -algebras and in this sense at least a deformation quantization is always guaranteed to exist. In addition they fit into the folklore, that observables should be the homology of something.

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## 2. SUMMARY

We suggest an  $n$ -plectic structure  $(A, \mathfrak{g}, \omega)$  to be a torsionless (B.4) Lie Rinehart pair  $(A, \mathfrak{g})$  with a degree  $(n+1)$  cocycle  $\omega$ , chosen from the de Rham complex  $\Omega^\bullet(\mathfrak{g}, A)$  of exterior cotensors (3.1).

The symplectic Poisson algebra of differentiable functions is then refined to the set of all de Rham cotensors  $f \in \Omega^\bullet(\mathfrak{g}, A)$ , which satisfy the constrained first order differential equation

$$i_x \omega = df \quad \text{and} \quad i_y \omega = f,$$

for some, not necessarily unique, Hamilton tensor  $x$  and Poisson constraint  $y$ .

We write  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  for this solution set (3.2) and call it the  $n$ -plectic Poisson algebra, since it is a homotopy Poisson- $n$  algebra (A.3) with respect to the following operations:

A differential graded commutative structure (3.7), inherited from the de Rham algebra  $\Omega^\bullet(\mathfrak{g}, A)$ . This means, that the differential restricts to a map

$$d : \mathcal{Pois}(A, \mathfrak{g}, \omega) \rightarrow \mathcal{Pois}(A, \mathfrak{g}, \omega)$$

on Poisson cotensors and that the ordinary exterior cotensor product restricts to an  $n$ -plectic multiplication

$$\wedge : \mathcal{Pois}(A, \mathfrak{g}, \omega) \times \mathcal{Pois}(A, \mathfrak{g}, \omega) \rightarrow \mathcal{Pois}(A, \mathfrak{g}, \omega) .$$

If  $x_1$  and  $x_2$  are Hamilton tensors associated to the (homogeneous) Poisson cotensors  $f_1$  and  $f_2$  by the fundamental equation  $i_x \omega = df$ , the exterior tensor

$$x_{f_1 \wedge f_2} := (-1)^{|f_1|} j_{f_1} x_2 + (-1)^{(|f_1|-1)|f_2|} j_{f_2} x_1$$

is a Hamilton tensor, associated to  $f_1 \wedge f_2$  by the same equation. Moreover, if  $y_2$  is a Poisson constraint associated to  $f_2$  by the constraint equation  $i_y \omega = f$ , the exterior tensor

$$y_{f_1 \wedge f_2} := j_{f_1} y_2$$

is a Poisson constraint, associated to  $f_1 \wedge f_2$ . (Here  $j_{(\cdot)}(\cdot)$  is the left contraction of a tensor by a cotensor (65)) This makes Poisson cotensors into a differential graded commutative algebra (3.7).

The first step beyond this commutative structure appears as the homotopy Poisson 2-bracket, which is defined for any Poisson cotensors  $f_1$  and  $f_2$  and associated (homogeneous) Hamilton tensors  $x_1$  and  $x_2$  by

$$\{f_1, f_2\} = -L_{x_1} f_2 + (-1)^{(|x_1|-1)(|x_2|-1)} L_{x_2} f_1 .$$

This bracket does not depend on the actual choices (3.11) of  $x_1$  or  $x_2$  and is  $(n-1)$ -fold graded antisymmetric with respect to the tensor degree of the  $f_i$ 's. The de Rham differential acts as a derivation (3.12) and an associated Hamilton tensor can be computed by the Schouten-Nijenhuis bracket

$$x_{\{f_1, f_2\}} = 2[x_2, x_1] .$$

If  $y_1$  resp.  $y_2$  are Poisson constraints associated to  $f_1$  resp.  $f_2$ , a Poisson constraint of their Poisson bracket is given by

$$y_{\{f_1, f_2\}} = [x_2, y_1] - (-1)^{(|x_1|-1)(|x_2|-1)} [x_1, y_2] .$$

In case  $n = 1$  this bracket is equal to the usual symplectic Poisson bracket, up to the factor '2' and for  $n \geq 2$  it refines the  $n$ -plectic Lie bracket found by Rogers [17], with an additional coboundary term (3.4).

Unlike the symplectic case, the Jacobi equation does not hold 'on the nose' as long as  $n > 1$ . Instead it is satisfied up to certain coboundary terms (3.13) only. These terms are controlled by what we call the homotopy Poisson 3-bracket, which is defined for any three Poisson cotensors  $f_1, f_2$  resp.  $f_3$  and associated Hamilton tensors  $x_1, x_2$  resp.  $x_3$  by the 'shuffle contractions'

$$\begin{aligned} \{f_1, f_2, f_3\} := & i_{[x_2, x_1]} f_3 - (-1)^{(|x_2|-1)(|x_3|-1)} i_{[x_3, x_1]} f_2 \\ & + (-1)^{(|x_1|-1)(|x_2|-1)+(|x_1|-1)(|x_3|-1)} i_{[x_3, x_2]} f_1. \end{aligned}$$

Again this does not depend on the actual choices of neither  $x_1, x_2$  nor  $x_3$  and associated Hamilton or Poisson constraint tensors can be computed explicit (3.16).

With additional operators however come additional Jacobi-like equations and to control them we need additional operators. The general pattern is then an infinite series of  $k$ -ary brackets for arbitrary integers  $k \in \mathbb{N}$ . We call these operators the homotopy Poisson  $k$ -brackets and define them inductively:

If  $f_1, \dots, f_{k+1}$  are Poisson cotensors and  $x_{\{f_1, \dots, f_k\}}$  is a Hamilton tensor associated to their homotopy Poisson  $k$ -bracket for some  $k$ , the homotopy Poisson  $(k+1)$ -bracket is the shuffled contraction

$$\{f_1, \dots, f_{k+1}\} := \sum_{\sigma \in Sh(k,1)} (-1)^{\sigma+k} e(s; sx_1, \dots, sx_{k+1}) i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k)}\}}} f_{\sigma(k+1)},$$

where  $|sx|$  just means  $|x| + 1$ ,  $Sh(i, j)$  is the set of  $(i, j)$ -shuffle permutations (A.1) and  $(-1)^{\sigma} e(\sigma; \cdot, \dots, \cdot)$  is the antisymmetric Koszul sign (A.2.1) of  $\sigma$ .

Again this map does not depend on the choice of any  $x_i$  and an associated Hamilton tensor as well as a Poisson constrain can be computed inductively (3.18).

All these Poisson brackets interact with each other and the de Rham differential in terms of a so called  $(n-1)$ -fold shifted homotopy Lie algebra (C). The  $(n-1)$ -fold shifting is due to the degree relation  $|x| = |f| + n$  between a Poisson cotensor  $f$  and any associated Hamilton tensor  $x$ . This in turn originates from the fundamental equation  $i_x \omega = df$ .

The interaction between the Poisson 2-bracket and the exterior product is controlled by the Leibniz equation, which does not hold 'strictly' as long as  $n > 1$ , but only up to certain correction terms.

These correction terms in turn are controlled by what we call the first Leibniz operator. It is defined for any three Poisson cotensors  $f_1, f_2$  resp.  $f_3$  and associated Hamilton tensors  $x_1, x_2$  resp.  $x_3$  by

$$\begin{aligned} \{f_1 \| f_2, f_3\} := & -i_{x_1} (f_2 \wedge f_3) + (-1)^{(|x_1|-1)(|x_{f_2 \wedge f_3}|-1)} i_{x_{f_2 \wedge f_3}} f_1 \\ & + (-1)^{|x_1||f_2|} f_2 \wedge (i_{x_1} f_3 - (-1)^{(|x_1|-1)(|x_3|-1)} i_{x_3} f_1) \\ & + (i_{x_1} f_2 - (-1)^{(|x_1|-1)(|x_2|-1)} i_{x_2} f_1) \wedge f_3 \end{aligned}$$

and does not depend on the actual choices of any  $x_i$ . An associated Hamilton tensor as well as a Poisson constrain can be computed explicit. It is graded symmetric with respect to the arguments on the right and interacts with the Poisson bracket and the de Rham differential in terms of the  $n$ -plectic Leibniz equation:

$$\begin{aligned} \{f_1, f_2 \wedge f_3\} = & \{f_1, f_2\} \wedge f_3 + (-1)^{(|x_1|-1)|f_2|} f_2 \wedge \{f_1, f_3\} \\ & + d\{f_1 \| f_2, f_3\} + \{df_1 \| f_2, f_3\} - (-1)^{|x_1|} \{f_1 \| df_2, f_3\} \\ & - (-1)^{|f_2|+|x_1|} \{f_1 \| f_2, df_3\}. \end{aligned}$$

The interaction between the higher Poisson brackets and the exterior product has no analog in the symplectic case. Nevertheless, higher 'Leibniz-like' equations appear (37). We say that such an equation holds 'strictly' or 'on the nose' if

$$\begin{aligned} \{f_1, \dots, f_k, f_{k+1} \wedge f_{k+2}\} &= \{f_1, \dots, f_k, f_{k+1}\} \wedge f_{k+2} \\ &+ (-1)^{(\sum_{i=1}^k |x_i| - 1)|f_{k+1}|} f_{k+1} \wedge \{f_1, \dots, f_k, f_{k+2}\}. \end{aligned}$$

However, in general this equation does not hold strictly, but again only up to certain correction terms, which we call the higher Leibniz operators.

Typesetting these operators (36) and all the additional equations (38), (44) is challenging, since explicit expression get quite complicated for  $k \gg 1$ .

The complete picture is that the  $(n-1)$ -fold shifted Lie algebra of higher Poisson brackets interacts with the differential graded structure and the various Leibniz operators in terms of a homotopy Poisson- $n$  algebra.

Finally it should be noted, that in an actual  $n$ -plectic setting all operators are trivial beyond a certain bound  $\mathcal{O}(n)$ , which means that for small  $n$ , only 'a few' operators are different from zero. Loosely speaking we can say that the more  $n$  deviates from 1, the more the structure deviates from being a Poisson algebra.

### 3. HIGHER SYMPLECTIC STRUCTURES

**3.1. N-plectic structures.** We define higher symplectic structures as torsionless Lie Rinehart pairs together with a distinguished cocycle from their de Rham complex. For an introduction to these terms look at appendix (B), or the references therein.

**Definition 3.1.** *Let  $(A, \mathfrak{g})$  be a torsionless (semi-reflexive) Lie Rinehart pair and  $\omega \in \Omega^\bullet(\mathfrak{g}, A)$  a degree  $(n+1)$  cocycle with respect to the de Rham differential. Then  $(A, \mathfrak{g}, \omega)$  is called an  **$n$ -plectic structure** and  $\omega$  is called its  **$n$ -plectic cocycle**.*

We do not distinguish between  $n$ -plectic cocycles that are degenerate on vectors and those that are not. For a *homotopy Poisson- $n$  algebra* to exist here, the fundamental pairing between functions and vector fields has to generalize to tensors and cotensors *in a range of degrees* and almost all  $n$ -plectic cocycles have a non trivial kernel on higher tensors. Whether they are degenerated on vectors or not. Unique pairings are an exception in the general  $n$ -plectic setting.

*Remark.* Of course there are  $n$ -plectic structures, which are non degenerate on vectors. If this becomes important, we could call them *non-degenerate*. For the general theory however, this distinction is irrelevant.

The following example shows that any torsionless Lie Rinehart pair gives rise to an  $n$ -plectic structure, no matter how degenerate its de Rham complex might be:

*Example 1* (Trivial  $n$ -plectic structure). Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair and  $\omega \in \Omega^{n+1}(\mathfrak{g}, A)$  the zero cocycle. Then  $(A, \mathfrak{g}, \omega)$  is called the **trivial**  $n$ -plectic structure.

Since the idea of " $n$ -plectic" has such a long historical prologue, it appears in a lot of flavors. The following examples just list a few:

*Example 2* (Symplectic manifold). Any symplectic manifold  $(M, \omega)$  is equivalently an 1-plectic structure  $(C^\infty(M), \mathfrak{X}(M), \omega)$  on the Lie Rinehart pair of smooth functions and vector fields over  $M$ .

*Example 3* (Presymplectic manifold). Any presymplectic manifold  $(M, \omega)$  is equivalently an 1-plectic structure  $(C^\infty(M), \mathfrak{X}(M), \omega)$  on the Lie Rinehart pair of smooth functions and vector fields over  $M$ .

*Example 4* (Multisymplectic fiber bundle). Let  $p : P \rightarrow M$  be a smooth fiber bundle of rank  $N$  over an  $n$ -dimensional manifold equipped with a closed and non-degenerate  $(n+1)$ -form  $\omega$  defined on the total space which is  $(n-1)$ -horizontal and admits an involutive and isotropic vector subbundle of the vertical bundle  $VP$  of codimension  $N$  and dimension  $Nn+1$ . (In [2], this is called a *multisymplectic fiber bundle*). Then the triple  $(C^\infty(P), \mathfrak{X}(P), \omega)$  is an  $n$ -plectic structure.

**3.2. Poisson cotensors.** Every symplectic manifold has a Lie bracket, which interacts with the 'dot'-product of smooth functions in terms of a Poisson algebra. The origin of this bracket is the fundamental and unique pairing

$$i_x \omega = df \quad (1)$$

between functions and vector fields, defined in terms of the non-degenerate symplectic 2-form.

In this section, we generalize the idea to the  $n$ -plectic setting and show that a certain constraint condition can replace the requirement of the pairing to be unique. This insight goes back to the work of Forger, Paufler and Römer [4].

Before we start, let's note the following important observations: For a general  $n$ -plectic structure a pairing like (1) makes sense in a range of cotensor degrees, not only in degree  $(n-1)$ . However the kernel of  $\omega$  is potentially non-trivial on tensors of arbitrary degrees and consequently the association

$$\text{tensor} \Leftrightarrow \text{cotensor}$$

via equation (1) is neither always defined nor unique. To some extent, this already happens in presymplectic geometry [1].

One way to handle this would be to restrict the theory exclusively to  $n$ -plectic cocycles that are non-degenerate on vectors and the pairing (1) only to vectors and appropriate  $(n-1)$  cotensors. This however comes with a lot of disadvantages:

If we restrict to non-degenerate cocycles only, we exclude important applications like contact- or presymplectic-manifolds. Moreover not every torsionless Lie Rinehart pair has non-degenerate cocycles and therefore the theory wouldn't be natural anymore.

If we allow degeneracy, but restrict the pairing to certain vectors and  $(n-1)$  cotensors, a homotopy **Poisson** structure can not arise from the de Rham differential and the exterior product. Therefore such a theory is eventually restricted to a bare homotopy Lie structure, which is not enough for some approaches to quantization.

Another consideration is that the Jacobi identity of the symplectic Lie bracket depend on properties that can't be expected in a general  $n$ -plectic setting. If the Poisson structure has to be replaced by a more general *homotopy* Poisson- $n$  structure, a combination of the defining structure equations (49) and the fundamental pairing (1) leads to the equation

$$i_y \omega = \sum_{i+j=k+1}^{i \geq 1} \sum_{s \in Sh(j, k-j)} \pm \{ \{ f_{s(1)}, \dots, f_{s(j)} \}, f_{s(j+1)}, \dots, f_{s(k)} \}$$

and to ensure the existence of a solution  $y$  to this equation, the important additional constraint equation

$$i_y \omega = f \quad (2)$$

is required to hold on all arguments in addition to the first pairing in (1).

We call (2) the **Poisson constraint** of the fundamental pairing (1), since it is the only constraint that has to be made on the solutions of (1) in order to form a homotopy Poisson- $n$  algebra.

*Remark.* This additional constraint equation is completely invisible in symplectic geometry, as it only appears implicitly. To see that, let  $\eta$  be the Poisson bivector field, associated to the symplectic form  $\omega$ . Then the bivector  $f \cdot \eta$  is a solution to (2) for any function  $f$  since

$$i_{(f \cdot \eta)}\omega = f .$$

*Remark.* To my knowledge, such a Poisson constraint appeared first in the work of Forger and Römer [4], where they introduced it to handle the previously mentioned ambiguity inherent in the fundamental pairing (1). In [4], solutions to both equations (1) and (2) are called *Poisson forms*.

Summarizing all this, we define the  $n$ -plectic equivalent to the Poisson algebra of smooth functions as the cotensor solution set of the fundamental pairing with its Poisson constraint:

**Definition 3.2** (The fundamental equation and its Poisson constraint). *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure,  $X^\bullet(\mathfrak{g}, A)$  the exterior tensor power and  $\Omega^\bullet(A, \mathfrak{g})$  the exterior cotensor power. The **fundamental equation** of the  $n$ -plectic structure is the first order (algebraic) differential equation*

$$i_x\omega = df, \tag{3}$$

*defined on the product  $X^\bullet(\mathfrak{g}, A) \times \Omega^\bullet(\mathfrak{g}, A)$ . If the pair  $(x, f)$  is a solution,  $x$  is called a **Hamilton tensor**,  $f$  is called a **Hamilton cotensor** and both are called **associated** to each other.*

*We write  $\text{Ham}(A, \mathfrak{g}, \omega)$  for the set of all Hamilton tensors and  $\text{Ham}^*(A, \mathfrak{g}, \omega)$  for the set of all Hamiltonian cotensors.*

*Moreover the **Poisson constraint** of the  $n$ -plectic structure is the algebraic equation*

$$i_y\omega = f, \tag{4}$$

*defined on the product  $X^\bullet(\mathfrak{g}, A) \times \text{Ham}^*(A, \mathfrak{g}, \omega)$ . If a pair  $(y, f)$  is a solution,  $f$  is called a **Poisson cotensor** and  $y$  is called a **Poisson constraint** associated to  $f$ . We write  $\text{Pois}(A, \mathfrak{g}, \omega)$  for the set of all Poisson cotensors.*

Neither of the previously defined sets is empty: Both, the fundamental equation and its Poisson constraint are linear in each argument and therefore the kernel of  $\omega$  and the zero cotensor are always a solution.

Each Poisson cotensor  $f \in \text{Pois}(A, \mathfrak{g}, \omega)$  has a not necessarily unique Hamilton tensor  $x$  associated to  $f$  by the equation  $i_x\omega = df$  and a not necessarily unique Poisson tensor  $y$  associated to  $f$  by the equation  $i_y\omega = f$ . We usually use the symbol  $x$  to indicate that the tensor is Hamilton and the symbol  $y$  to indicate that the tensor is a Poisson constraint.

**Corollary 3.3.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $x \in \text{Ham}(A, \mathfrak{g}, \omega)$  a Hamilton tensor. Then  $L_x\omega = 0$ .*

*Proof.* Since  $i_x\omega = df$  for some cotensor  $f$  we get  $di_x\omega = 0$  and therefore  $L_x\omega = 0$  from Cartan's infinitesimal homotopy formula, since  $\omega$  is a cocycle.  $\square$

**Corollary 3.4.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure,  $f \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  a homogeneous Poisson cotensor with associated Poisson constraint  $y \in X(\mathfrak{g}, A)$  and associated Hamilton tensor  $x \in \mathcal{H}am(A, \mathfrak{g}, \omega)$ . If both  $x$  and  $y$  are homogeneous, then*

$$|x| = |f| + n \quad \text{and} \quad |y| = |x| + 1 \quad (5)$$

*with respect to the tensor grading. Moreover the following boundary equation is satisfied:*

$$di_y \omega = i_x \omega. \quad (6)$$

*Proof.* Immediate, since we assume tensors to be concentrated in positive tensor degrees and cotensors as concentrated in negative tensor degrees.  $\square$

One of the most important consequence of the Poisson constraint (4) is what we call the **kernel property**: The kernel of the  $n$ -plectic cocycle  $\omega$  is always contained in the kernel of a Poisson cotensor. As a consequence, the contractions of a Poisson cotensor along tensors, associated to the same cotensor, are equal. The following proposition makes this precise:

**Proposition 3.5.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $f \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  a Poisson cotensor. Then*

$$\ker(\omega) \subset \ker(f). \quad (7)$$

*If  $y$  and  $y'$  are tensors, with  $i_y \omega = f$  and  $i_{y'} \omega = f$ , the difference  $y - y'$  is an element of the kernel of  $\omega$  and the contractions  $i_y g$  and  $i_{y'} g$  are equal for all Poisson cotensors  $g \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$ .*

*Similar, if  $x$  and  $x'$  are tensors, with  $i_x \omega = df$  and  $i_{x'} \omega = df$ , the difference  $x - x'$  is an element of the kernel of  $\omega$  and the contractions  $i_x g$  and  $i_{x'} g$  are equal for all Poisson cotensors  $g \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$ .*

*Proof.* The first part is an implication of the Poisson constraint equation (4). To see that assume  $\xi \in \ker(\omega)$ . Then there is an exterior tensor  $y$  with  $i_\xi f = i_\xi i_y \omega = \pm i_y i_\xi \omega = 0$ .

For the second and third part compute  $0 = f - f = i_y \omega - i_{y'} \omega = i_{(y-y')} \omega$  and  $0 = df - df = i_x \omega - i_{x'} \omega = i_{(x-x')} \omega$ , respectively. Similar  $i_y g - i_{y'} g = i_{(y-y')} g = 0$  and  $i_x g - i_{x'} g = i_{(x-x')} g = 0$  follows from the kernel property of  $g$ .  $\square$

**3.3. The differential graded commutative algebra.** We show that the exterior product and the de Rham differential project from arbitrary cotensors to Poisson cotensors and derive explicit formulas for their associated Hamilton tensors and Poisson constraints. This refines the symplectic function algebra to the general  $n$ -plectic setting.

We start with an important technical detail, which is a consequence of the Poisson constraint equation (4) and equation (72):

**Proposition 3.6.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure,  $f \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  a Poisson cotensor and  $x \in X(\mathfrak{g}, A)$  any tensor. Then*

$$i_{j_f x} \omega = f \wedge i_x \omega. \quad (8)$$

*Proof.* Since  $f$  is Poisson, there is an associated Poisson constraint  $y \in X^\bullet(\mathfrak{g}, A)$  with  $i_y \omega = f$ . Equation (72) then implies the proposition.  $\square$

Equation (8) does not work for arbitrary cotensors. As the following counterexample shows, the cotensor really needs to be Poisson:

*Example 5.* Consider the presymplectic manifold  $(\mathbb{R}^3, dx^1 \wedge dx^2)$ , the closed 1-form  $f := dx^3$  and the Hamilton vector field  $X := \partial_1$ . Clearly there can not be a vector field  $Y$  such that  $i_Y \omega = dx^3$  and therefore  $f$  is Hamilton but not Poisson. In this case equation (8) is not satisfied since we compute

$$i_{j_f X} \omega = i_{j_{dx^3} \partial_1} \omega = 0 \neq dx^3 \wedge dx^2 = f \wedge i_X \omega .$$

Besides the kernel property (3.5), this simple counterexample is another justification for our definition of Poisson cotensors. The existence of a Poisson constraint (2), hence a valid equation (8), is necessary for the exterior product to close on Poisson cotensors as we show in (3.7).

The following theorem is the first major step towards our homotopy Poisson- $n$  algebra. It solves the long standing problem of how to define a natural product in higher symplectic geometry. In a subsequent corollary we give explicit constructions for associated Hamilton tensors and Poisson constraints.

**Theorem 3.7.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure. Then  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  is a differential graded commutative and associative subalgebra of the de Rham algebra  $\Omega^\bullet(\mathfrak{g}, A)$ .*

*Proof.* We have to show, that  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  is a linear subspace and that both the exterior product and the de Rham differential close on Poisson cotensors, i.e. that the fundamental equation (3) and its Poisson constraint equation (4) have solutions.

To see the linear structure, observe that both the fundamental equation and its Poisson constraint equation (4) are linear in all arguments.

To see that the exterior differential closes, let  $f \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  be a Poisson cotensor. Then there exists a Hamilton tensor  $x$ , associated to  $f$  by the fundamental equation  $i_x \omega = df$ , which in turn is a Poisson constraint associated to  $df$ . In addition any element from  $\ker(\omega)$  is a Hamilton tensor associated to  $df$ . Therefore  $df \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$

To solve the Poisson constraint equation (4), for the exterior product  $f_1 \wedge f_2$  of Poisson cotensors  $f_1$  and  $f_2 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$ , let  $y \in X^\bullet(\mathfrak{g}, A)$  be a Poisson constraint associated to  $f_2$ . Using (8) we get  $i_{j_{f_1} y} \omega = f_1 \wedge f_2$ , which shows that  $j_{f_1} y$  is a Poisson tensor associated to  $f_1 \wedge f_2$ .

To find a solution to the fundamental equation (1), let  $x_1$  and  $x_2$  be solutions of  $i_{x_1} \omega = df_1$  and  $i_{x_2} \omega = df_2$ , respectively, which exists, since both cotensors are Poisson. Then, using (8), we compute

$$\begin{aligned} & (-1)^{|f_1|} i_{j_{f_1} x_2} \omega + (-1)^{(|f_1|-1)|f_2|} i_{j_{f_2} x_1} \omega \\ &= (-1)^{|f_1|} f_1 \wedge df_2 + (-1)^{(|f_1|-1)|f_2|} f_2 \wedge df_1 \\ &= df_1 \wedge f_2 + (-1)^{|f_1|} f_1 \wedge df_2 \\ &= d(f_1 \wedge f_2) . \end{aligned}$$

□

The previous proof provides an explicit way to compute Hamilton tensors and Poisson constraints associated to the exterior product of two Poisson cotensors. In particular we have:



**Corollary 3.8** (Associated tensors). *Let  $f_1, f_2 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  be two Poisson cotensors with associated Hamilton tensors  $x_1$  and  $x_2$ , respectively. Then the exterior tensor*

$$x_{f_1 \wedge f_2} := (-1)^{|f_1|} j_{f_1} x_2 + (-1)^{(|f_1|-1)|f_2|} j_{f_2} x_1 \quad (9)$$

*is a solution to the fundamental equation  $i_{x_{f_1 \wedge f_2}} \omega = d(f_1 \wedge f_2)$  and therefore a Hamilton tensor associated to  $f \wedge g$ .*

*If  $y_2$  is associated to  $f_2$  by the Poisson constraint equation  $i_{y_2} \omega = f_2$ , the exterior tensor*

$$y_{f_1 \wedge f_2} := j_{f_1} y_2 \quad (10)$$

*is a solution to the equation  $i_{y_{f_1 \wedge f_2}} \omega = f_1 \wedge f_2$  and therefore a Poisson constraint associated to  $f \wedge g$ .*

*Proof.* See the proof of theorem (3.7).  $\square$

*Remark.* Strictly speaking it is not correct to write  $x_{f_1 \wedge f_2}$  since associated tensors are in general not unique and the expression depends on the particular chosen Hamilton tensors  $x_i$ . By abuse of notation we stick to the symbol  $x_{f_1 \wedge f_2}$  to indicate that we have at least *some* Hamilton tensor associated to the exterior product  $f_1 \wedge f_2$ . According to proposition (3.5) all representatives compute equal contractions and Lie derivations of Poisson cotensors and therefore the risk of confusion is low.

**Corollary 3.9.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure. Then the differential graded algebra  $(\mathcal{Pois}(A, \mathfrak{g}, \omega), \wedge, d)$ , satisfies the structure equation (4.9) of a homotopy Poisson- $n$  algebra for the parameter  $k = 1$  and all  $p_1 \in \mathbb{N}$ .*

*Proof.* Define the needed structure maps of (4.8) for  $k = 1$  and  $f_1, f_2 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  by:  $D_1(s^{n-1} f_1) := s d f_1$ ,  $D_2(s^{n-2}(\overline{s f_1 \otimes s f_2})) := (-1)^{|s f_1|} s(f_1 \wedge f_2)$  and  $D_q = 0$  for all  $q \geq 3$ .

Then all of these shifted structure maps are homogeneous of degree  $(1 - n)$  and since  $D_2(s^{n-2}(\overline{s f_2 \otimes s f_1})) + (-1)^{|s f_1| |s f_2|} D_2(s^{n-2}(\overline{s f_1 \otimes s f_2})) = 0$ , they have the correct (shifted) Harrison symmetry.

Since  $D_q = 0$  for all  $q \geq 3$ , equation (4.9) is only non trivial for  $p_1 \leq 3$ . For  $p_1 = 1$  it is the 'square zero condition' on the differential, for  $p_1 = 2$  it is the requirement, that  $d$  is a derivation with respect to the exterior product and for  $p_1 = 3$  it is the associativity law. All this follows from the differential graded structure on  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$ .  $\square$

**3.4. The Poisson Bracket.** We refine the symplectic Poisson bracket to the general  $n$ -plectic setting. This bracket has all the expected properties, except that for  $n \geq 2$ , neither the Jacobi nor the Leibniz identity is satisfied strictly, but in terms of a *homotopy* Poisson- $n$  algebra.

The following approach was inspired by the work of Forger and Römer [4], but works without the additional coboundary condition on the  $n$ -plectic cocycle. In contrast to their bracket, this one interacts accurate with the exterior derivative. In addition it can be seen as some kind of "Poisson-theoretic" refinement of the  $n$ -plectic homotopy *Lie* bracket as given by Rogers in [17].

**Definition 3.10.** Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  the set of Poisson cotensors. The map

$$\{\cdot, \cdot\} : \mathcal{Pois}(A, \mathfrak{g}, \omega) \times \mathcal{Pois}(A, \mathfrak{g}, \omega) \rightarrow \mathcal{Pois}(A, \mathfrak{g}, \omega), \quad (11)$$

defined for any homogeneous Poisson cotensors  $f_1, f_2 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and associated Hamilton tensors  $x_1, x_2 \in \mathcal{Ham}(A, \mathfrak{g}, \omega)$  by the equation

$$\{f_1, f_2\} = -L_{x_1}f_2 + (-1)^{(|x_1|-1)(|x_2|-1)}L_{x_2}f_1 \quad (12)$$

and then extended to all of  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  by linearity, is called the **homotopy Poisson 2-bracket**.

We propose the 'homotopy' modifier since theorem (3.13) and (3.21) show that neither the Jacobi nor the Leibniz identity hold 'strictly' but only 'up to higher homotopies' as we will see in the next sections. The bracket is the antisymmetric incarnation of the one I developed in [15]. It interacts with the differential and the exterior product in terms of a so called *homotopy Poisson- $n$  algebra* (??), which justifies the name.

*Remark.* To highlight the relation to the  $n$ -plectic homotopy Lie bracket as defined by Rogers in [17], observe the identity

$$\frac{1}{2}\{f_1, f_2\} = (-1)^{|x_1|} \cdot i_{x_2 \wedge x_1} \omega - \frac{1}{2}d(i_{x_1}f_2 - (-1)^{(|x_1|-1)(|x_2|-1)}i_{x_2}f_1). \quad (13)$$

The additional cocycle can be seen as some sort of Poisson-theoretic correction term, necessary for a correct interaction with the differential graded commutative structure.

*Remark* (Symplectic manifolds). Let  $(M, \omega)$  be a symplectic manifold. The homotopy Poisson 2-bracket is equal to the common symplectic Poisson bracket, usually defined by  $-i_{x_1 \wedge x_2} \omega$  (up to the factor 2). This is a consequence of the previous remark. In particular the Jacobi identity holds strictly for  $n = 1$ .

On the technical level, a first thing to show is, that the bracket is independent of the particular chosen associated Hamilton tensors. This is guaranteed by the kernel property (3.5). The following proposition gives the details and computes associated tensors explicitly.

**Proposition 3.11.** Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $f_1, f_2 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  two Poisson cotensors. The image  $\{f_1, f_2\}$  is a well defined Poisson cotensor. If  $y_1$  resp.  $y_2$  and  $x_1$  resp.  $x_2$  are tensors associated to  $f_1$  and  $f_2$  by the equations  $i_{x_i} \omega = df_i$  and  $i_{y_i} \omega = f_i$ , respectively, the exterior tensor

$$y_{\{f_1, f_2\}} = [x_2, y_1] - (-1)^{(|x_1|-1)(|x_2|-1)}[x_1, y_2]$$

is associated to the homotopy Poisson 2-bracket by the equation  $i_{y_{\{f_1, f_2\}}} \omega = \{f_1, f_2\}$  and the exterior tensor

$$x_{\{f_1, f_2\}} := [x_2, x_1] - (-1)^{(|x_1|-1)(|x_2|-1)}[x_1, x_2] \quad (14)$$

is associated to the homotopy Poisson 2-bracket by the equation  $i_{x_{\{f_1, f_2\}}} \omega = d\{f_1, f_2\}$ .

*Proof.* To see that  $\{f_1, f_2\}$  is well defined, suppose  $\xi$  is a tensor from the kernel of  $\omega$ . Then  $L_\xi f = di_\xi f - (-1)^{|\xi|}i_\xi df = 0$  since  $f$  and  $df$  have the kernel property. Therefore we get  $L_{x+\xi}f = L_x f$  for any Poisson cotensor  $f$  and Hamilton tensor  $x$ . By prop. (3.5) the difference of tensors, associated to the same Poisson cotensor is

an element of the kernel of  $\omega$  and consequently the image  $\{f_1, f_2\}$  does not depend on any particular choice.

To show that  $y_{\{f_1, f_2\}}$  satisfies the Poisson constraint equation with respect to the homotopy Poisson 2-bracket, we use  $L_{x_i}\omega = 0$  as well as  $|x_i| = |y_i| - 1$  and compute:

$$\begin{aligned}
i_{y_{\{f_1, f_2\}}}\omega &= -(-1)^{(|x_1|-1)(|x_2|-1)}i_{[x_1, y_2]}\omega + i_{[x_2, y_1]}\omega \\
&= -(-1)^{(|x_1|-1)(|x_2|-1)}\left((-1)^{(|x_1|-1)(|x_2|-1)}L_{x_1}i_{y_2}\omega - i_{y_2}L_{x_1}\omega\right) \\
&\quad + \left((-1)^{(|x_1|-1)(|x_2|-1)}L_{x_2}i_{y_1}\omega - i_{y_1}L_{x_2}\omega\right) \\
&= -L_{x_1}i_{y_2}\omega + (-1)^{(|x_1|-1)(|x_2|-1)}L_{x_2}i_{y_1}\omega \\
&= -L_{x_1}f_2 + (-1)^{(|x_1|-1)(|x_2|-1)}L_{x_2}f_1.
\end{aligned}$$

It only remains to show that the tensor  $x_{\{f_1, f_2\}}$  satisfies the fundamental equation with respect to the homotopy Poisson 2-bracket:

$$\begin{aligned}
&i_{[x_2, x_1]}\omega - (-1)^{(|x_1|-1)(|x_2|-1)}i_{[x_1, x_2]}\omega \\
&= ((-1)^{(|x_2|-1)|x_1|}L_{x_2}i_{x_1}\omega - i_{x_1}L_{x_2}\omega) \\
&\quad - (-1)^{(|x_1|-1)(|x_2|-1)}((-1)^{(|x_1|-1)|x_2|}L_{x_1}i_{x_2}\omega - i_{x_2}L_{x_1}\omega) \\
&= -(-1)^{(|x_1|-1)(|x_2|-1)+|x_2|}L_{x_2}df_1 + (-1)^{|x_1|}L_{x_1}df_2 \\
&= (-1)^{(|x_1|-1)(|x_2|-1)}dL_{x_2}f_1 - dL_{x_1}f_2 \\
&= d((-L_{x_1}f_2 + (-1)^{(|x_1|-1)(|x_2|-1)}L_{x_2}f_1)) \\
&= d\{f_1, f_2\}.
\end{aligned}$$

□

*Remark.* Strictly speaking it is not correct to write  $x_{\{f_1, f_2\}}$  since associated tensors are in general not unique and the expression depends on the particular chosen Hamiltons  $x_i$ . By abuse of notation we stick to the symbol  $x_{\{f_1, f_2\}}$  to indicate that we have at least *some* Hamilton tensor associated to the Poisson 2-bracket  $\{f_1, f_2\}$ . According to proposition (3.5) all representatives compute equal contractions and Lie derivations of Poisson cotensors, therefore the risk of confusion is low.

Now lets see that the  $n$ -plectic Poisson 2-bracket has the properties expected from the bilinear bracket operator of a homotopy Poisson- $n$  algebra:

**Proposition 3.12.** *The homotopy Poisson 2-bracket is bilinear and homogeneous of degree  $(n-1)$  with respect to the tensor grading. It satisfies the  $(n-1)$ -fold shifted antisymmetry equation*

$$\{f_1, f_2\} = -(-1)^{(|f_1|+n-1)(|f_2|+n-1)}\{f_2, f_1\} \quad (15)$$

for all homogeneous  $f_1, f_2 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and the  $(n-1)$ -fold shifted homotopy Jacobi equation in dimension two

$$d\{f_1, f_2\} = \{df_1, f_2\} - (-1)^{(|f_1|+n-1)(|f_2|+n-1)}\{df_2, f_1\}. \quad (16)$$

*Proof.* Bilinearity is a straight forward implication of the definition, since all relevant operators are linear in all arguments. We assume that  $f_1, f_2 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$

are homogeneous. Since  $|f| = |x| - n$ , homogeneity can be computed by

$$\begin{aligned} |\{f_1, f_2\}| &= |L_{x_1} f_2| \\ &= |x_1| + |f_2| - 1 \\ &= |f_1| + |f_2| + n - 1. \end{aligned}$$

Similar, the  $(n-1)$ -fold shifted antisymmetry is a consequence of  $|f| = |x| - n$  and can be computed according to

$$\begin{aligned} &\{f_1, f_2\} \\ &= -L_{x_1} f_2 + (-1)^{(|x_1|-1)(|x_2|-1)} L_{x_2} f_1 \\ &= -(-1)^{(|x_1|-1)(|x_2|-1)} (-L_{x_2} f_1 + (-1)^{(|x_1|-1)(|x_2|-1)} L_{x_1} f_2) \\ &= -(-1)^{(|x_1|-1)(|x_2|-1)} \{f_2, f_1\} \\ &= -(-1)^{(|f_1|+n-1)(|f_2|+n-1)} \{f_2, f_1\}. \end{aligned}$$

Finally we compute the  $(n-1)$ -fold shifted homotopy Jacobi equation in dimension two. Since any Hamilton tensor associated to the cocycle  $df$  is an element from the kernel of  $\omega$ , we get

$$\begin{aligned} d\{f_1, f_2\} &= -dL_{x_1} f_2 + (-1)^{(|x_1|-1)(|x_2|-1)} dL_{x_2} f_1 \\ &= -(-1)^{|x_1|-1} L_{x_1} df_2 - (-1)^{(|x_1|-1)(|x_2|-1)+|x_2|} L_{x_2} df_1 \\ &= +(-1)^{(|x_1|-2)(|x_2|-1)} L_{x_2} df_1 - (-1)^{|x_1|-1} L_{x_1} df_2 \\ &= \{df_1, f_2\} - (-1)^{(|x_1|-1)(|x_2|-1)} \{df_2, f_1\} \\ &= \{df_1, f_2\} - (-1)^{(|f_1|+n-1)(|f_2|+n-1)} \{df_2, f_1\}. \end{aligned}$$

□

For  $n \geq 2$ , the  $n$ -plectic Poisson bracket does not satisfy the usual Jacobi identity. Still this 'failure' is not random, but controlled by a certain cocycle, that emerge from an additional 3-ary bracket, as we will see in the next section. The corrected Jacobi identity is equation (21). It is first evidence for the appearance of a homotopy algebra in the  $n$ -plectic formalism.

**Theorem 3.13** (Jacobi Identity). *The Poisson 2-bracket does not satisfy the Jacobi identity of a graded Lie algebra. Instead*

$$\begin{aligned} &\sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{\{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)}\} \\ &= - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) L_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)} \end{aligned} \quad (17)$$

for any homogeneous Poisson cotensors  $f_1, f_2, f_3 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and associated Hamilton tensors  $x_1, x_2$  and  $x_3$ , respectively.

*Proof.* To show that the equation is satisfied, apply the definition of  $\{\cdot, \cdot\}$  to rewrite the left side into

$$\begin{aligned} &- \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) L_{2 \cdot [x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)} \\ &+ \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) (-1)^{(|[x_{\sigma(2)}, x_{\sigma(1)}]|+1)(|x_{\sigma(3)}|+1)} L_{x_{\sigma(3)}} \\ &\quad \left( -L_{x_{\sigma(1)}} f_{\sigma(2)} + (-1)^{(|x_{\sigma(1)}|+1)(|x_{\sigma(2)}|+1)} L_{x_{\sigma(2)}} f_{\sigma(1)} \right). \end{aligned}$$

Since all terms in this expression are graded antisymmetric, we can simplify and reorder to arrive at:

$$\begin{aligned} & -2 \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) L_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)} \\ & + \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \cdot \\ & \cdot ((-1)^{(|x_{\sigma(1)}|+1)(|x_{\sigma(2)}|+1)} L_{x_{\sigma(2)}} L_{x_{\sigma(1)}} f_{\sigma(3)} - L_{x_{\sigma(1)}} L_{x_{\sigma(2)}} f_{\sigma(3)}) . \end{aligned}$$

Using (69) the second shuffle sum can be rewritten in terms of the Schouten bracket to get the expression

$$\begin{aligned} & -2 \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) L_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)} \\ & + \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) L_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)} \end{aligned}$$

which proof the theorem on homogeneous and therefore on arbitrary Poisson cotensors.  $\square$

In contrast to the  $n$ -plectic Poisson bracket, the Schouten-Nijenhuis bracket does satisfy the usual Jacobi equation. From this follows, that the  $n$ -plectic Jacobi identity, although not trivial, is nevertheless a cocycle:

**Theorem 3.14.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure. The Jacobi expression*

$$\sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{ \{ f_{\sigma(1)}, f_{\sigma(2)} \}, f_{\sigma(3)} \}$$

*is a cocycle for any three Poisson cotensors  $f_1, f_2, f_3 \in \text{Pois}(A, \mathfrak{g}, \omega)$ . If  $y_1, y_2, y_3$  are Poisson constraints, associated to  $f_1, f_2$  and  $f_3$ , then the exterior tensor given by*

$$- \sum_{\sigma \in Sh(1,2)} e(\sigma; sx_1, sx_2, sx_3) [[x_{\sigma(3)}, x_{\sigma(2)}], y_{\sigma(1)}] \quad (18)$$

*is Poisson constraint associated to the Jacobi expression by equation (4).*

*Proof.* To see that the Jacobi expression is a cocycle, we use the ordinary Jacobi identity of the Schouten bracket for exterior tensors. In particular we compute

$$\begin{aligned} 0 &= \sum_{\sigma \in Sh(2,1)} e(\sigma; sx_1, sx_2, sx_3) i_{[[x_{\sigma(3)}, x_{\sigma(2)}], x_{\sigma(1)}]} \omega \\ &= \sum_{\sigma \in Sh(2,1)} e(\sigma; sx_1, sx_2, sx_3) i_{[[x_{\sigma(2)}, x_{\sigma(1)}], x_{\sigma(3)}]} \omega \\ &= - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) (-1)^{|x_{\sigma(1)}|+|x_{\sigma(2)}|} L_{[x_{\sigma(2)}, x_{\sigma(1)}]} i_{x_{\sigma(3)}} \omega \\ &= - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) (-1)^{|x_{\sigma(1)}|+|x_{\sigma(2)}|} L_{[x_{\sigma(2)}, x_{\sigma(1)}]} df_{\sigma(3)} \\ &= - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) dL_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)} \\ &= d \left( \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{ \{ f_{\sigma(1)}, f_{\sigma(2)} \}, f_{\sigma(3)} \} \right) . \end{aligned}$$

$\square$

The existence of the associated tensor (18) is tied to the assumption that any Poisson cotensor  $f$  has a solution  $y$  to the Poisson constraint equation  $i_y \omega = f$ . If we drop that assumption, the equation

$$i_y \omega = - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{ \{ f_{\sigma(1)}, f_{\sigma(2)} \}, f_{\sigma(3)} \} \quad (19)$$

must not have a solution anymore, as the following simple counterexample shows:

*Example 6.* Let  $(\mathbb{R}^6, \omega)$  be the 3-plectic manifold, with 3-plectic cocycle given by

$$\omega := dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^6 + dx^2 \wedge dx^4 \wedge dx^5 \wedge dx^6 .$$

and consider  $f_1 := (x_1^2 x_3 - x_4) dx^5 \wedge dx^6$  and  $f_2 := -(x_3 + x_2^2 x_4) dx^5 \wedge dx^6$ . These differential form are Poisson, since Poisson constraint tensor fields are given (for example) by

$$y_1 := (x_1^2 x_3 - x_4) \partial_3 \wedge \partial_1 \quad , \quad y_2 := -(x_3 + x_2^2 x_4) \partial_4 \wedge \partial_2$$

and Hamilton tensor fields associated by the fundamental equation (1) are given (for example) by

$$x_1 := x_1^2 \partial_1 - \partial_2 - 2x_1 x_3 \partial_3 \quad , \quad x_2 := -\partial_1 - x_2^2 \partial_2 + 2x_2 x_4 \partial_4 .$$

In this case the Schouten bracket reduces to the usual Lie bracket and is given by

$$[x_2, x_1] = 2x_1 \partial_1 + 2x_2 \partial_2 - 2x_3 \partial_3 - 2x_4 \partial_4 .$$

Next define the differential form  $f_3 := dx^1 \wedge dx^2$ . Since any closed form is a Hamilton cotensor, so is  $f_3$ . In contrast  $f_3$  is not Poisson, because there can't be any tensor field  $y$  satisfying  $i_y \omega = f_3$ .

To find the Jacobian  $\sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{\{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)}\}$  we compute along the line of the proof of equation (3.13). Since  $f_3$  is a cocycle, this gives  $L_{[x_2, x_1]} f_3$ , which is  $4dx^1 \wedge dx^2$ . It follows that the Jacobi expression is not Poisson and that equation (19) has no solution.

Again, this justifies our proposed definition of Poisson cotensors. If we want a trilinear bracket operator to be related to our Poisson 2-bracket by the homotopy Jacobi equation in dimension four, a solution to the equation

$$\begin{aligned} i_{x_{\{f_1, f_2, f_3\}}} \omega = & - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{\{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)}\} \\ & - \sum_{\sigma \in Sh(1,2)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{df_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}\} \end{aligned}$$

is required and since the contraction operator is linear, this needs a solution of equation (19).

*Remark.* The Leibniz equation is in general not satisfied 'strictly' by the  $n$ -plectic Poisson bracket, but only up to certain corrections terms. We will look at this in detail in section (3.6). The correct homotopy Leibniz equation is then derived in theorem (3.21).

**3.5. The higher Poisson brackets.** Summarizing the last two sections we have seen that the general  $n$ -plectic setting has something close to a Poisson algebra, but without the Jacobi and the Leibniz identity.

In this section we look in detail on *how* the Jacobi equation fails. As it turns out this "failure" is controlled by another bracket-like operator that has three arguments and is therefore called the Poisson 3-bracket.

With additional operators, however, come additional Jacobi-like equations and to control them we need additional operators. The general pattern is then an infinite series of  $k$ -ary brackets for arbitrary integers  $k \in \mathbb{N}$ , that interact in terms of a so called  $(n-1)$ -fold shifted homotopy Lie algebra.

From a practical point of view it should be noted, that in an actual  $n$ -plectic setting all these  $k$ -ary brackets are trivial beyond a certain bound  $\mathcal{O}(n)$ , which means that for small  $n$  only 'a few' brackets are different from zero. Loosely speaking

we can say that the more  $n$  deviates from 1, the more the structure deviates from being a Poisson algebra.

**Definition 3.15.** Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  the set of Poisson cotensors. The **homotopy Poisson 3-bracket**

$$\{\cdot, \cdot, \cdot\} : \times^3 \mathcal{Pois}(A, \mathfrak{g}, \omega) \rightarrow \mathcal{Pois}(A, \mathfrak{g}, \omega) \quad (20)$$

is defined for any homogeneous  $f_1, f_2, f_3 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and associated Hamilton tensors  $x_1, x_2$  resp.  $x_3$ , by the equation

$$\{f_1, f_2, f_3\} = \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) i_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)}$$

and is then extended to all of  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  by linearity.

Again the kernel property (3.5) guarantees that this definition does not depend on the actual choice of any associated tensors. Hamilton tensors and Poisson constraints can be computed explicitly.

**Theorem 3.16.** The homotopy Poisson 3-bracket is trilinear, well defined and homogeneous of degree  $(2n - 1)$ . It is  $(n - 1)$ -fold shifted graded antisymmetric and for any three homogeneous Poisson cotensors  $f_1, f_2, f_3 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  the  $(n - 1)$ -fold shifted homotopy Jacobi equation in dimension four

$$\begin{aligned} d\{f_1, f_2, f_3\} + \sum_{\sigma \in Sh(1,2)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{df_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)}\} \\ + \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{\{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)}\} = 0 \end{aligned} \quad (21)$$

is satisfied. If  $y_1, y_2, y_3$  are Poisson constraints and  $x_1, x_2$  and  $x_3$  are Hamilton tensors, associated to  $f_1, f_2$  and  $f_3$  respectively, the tensor

$$y_{\{f_1, f_2, f_3\}} := \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) y_{\sigma(3)} \wedge [x_{\sigma(2)}, x_{\sigma(1)}] \quad (22)$$

is a Poisson constraints associated to the homotopy Poisson 3-bracket. In addition an associated Hamilton tensor is given by

$$\begin{aligned} x_{\{f_1, f_2, f_3\}} := \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) [[x_{\sigma(3)}, x_{\sigma(2)}], y_{\sigma(1)}] \\ - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) (-1)^{|x_{\sigma(1)}| + |x_{\sigma(2)}|} x_{\sigma(3)} \wedge [x_{\sigma(2)}, x_{\sigma(1)}]. \end{aligned}$$

*Proof.* To see that the definition does not depend on the particular chosen associated Hamilton tensors, we use (3.5) and proceed as in the proof of (3.11).

If we assume that  $f_1, f_2$  and  $f_3$  are homogeneous, we can see the homogeneity of the Poisson 3-bracket, for example by

$$\begin{aligned} |\{f_1, f_2, f_3\}| &= |i_{[x_2, x_1]} f_3| \\ &= |x_1| + |x_2| + |f_3| - 1 \\ &= |f_1| + |f_2| + |f_3| - 1 + 2n. \end{aligned}$$

Since  $[x_i, x_j] = -(-1)^{(|f_i|+n-1)(|f_j|+n-1)} [x_j, x_i]$ , the  $(n - 1)$ -fold shifted graded antisymmetry follows directly from the definition of the Poisson 3-bracket.

To compute the homotopy Jacobi equation in dimension four, we apply the definition of the differential and the trinary bracket to rewrite the mostleft term of the identity

$$d\{f_1, f_2, f_3\} = \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) di_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)}$$

and insert appropriate correction terms, using  $||[x_i, x_j]|| = |x_i| + |x_j| + 1$ . This leads to

$$\begin{aligned} & \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \cdot \\ & \quad \cdot (di_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)} + (-1)^{|x_{\sigma(1)}| + |x_{\sigma(2)}|} i_{[x_{\sigma(2)}, x_{\sigma(1)}]} df_{\sigma(3)}) \\ & - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) (-1)^{|x_{\sigma(1)}| + |x_{\sigma(2)}|} i_{[x_{\sigma(2)}, x_{\sigma(1)}]} df_{\sigma(3)}. \end{aligned}$$

Now we can rewrite the first shuffle sum, using Cartans graded infinitesimal omo-  
topy formular, into a sum over Lie derivations:

$$\begin{aligned} & \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) L_{[x_{\sigma(2)}, x_{\sigma(1)}]} f_{\sigma(3)} \\ & - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) (-1)^{|x_{\sigma(1)}| + |x_{\sigma(2)}|} i_{[x_{\sigma(2)}, x_{\sigma(1)}]} df_{\sigma(3)}. \end{aligned}$$

According to (3.13) the first shuffle sum is just the negative Jacobi expression. Since any Hamilton tensor associated to  $df_{\sigma(3)}$  must be an element of the kernel of  $\omega$ , we can rewrite the last expression to arrive at the right side of the equation:

$$\begin{aligned} & - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{ \{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)} \} \\ & - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) (-1)^{|x_{\sigma(1)}| + |x_{\sigma(2)}|} \{f_{\sigma(1)}, f_{\sigma(2)}, df_{\sigma(3)}\} = \\ & - \sum_{\sigma \in Sh(2,1)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{ \{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)} \} \\ & \quad - \sum_{\sigma \in Sh(1,2)} (-1)^\sigma e(\sigma; sx_1, sx_2, sx_3) \{ df_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)} \}. \end{aligned}$$

To see that  $y_{\{f_1, f_2, f_3\}}$  is a Poisson constraint associated to the trinary bracket, apply the contraction of  $\omega$  along  $y_{\{f_1, f_2, f_3\}}$  using  $i_{y_i} \omega = f_i$ .

To see that  $x_{\{f_1, f_2, f_3\}}$  is a Hamilton tensor associated to  $\{f_1, f_2, f_3\}$  use (18) to compute:

$$\begin{aligned} i_{x_{\{f_1, f_2, f_3\}}} \omega &= - \sum_{s \in Sh(2,1)} (-1)^\sigma e(s; sx_1, sx_2, sx_3) \{ \{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)} \} \\ & - \sum_{s \in Sh(2,1)} (-1)^\sigma e(s; sx_1, sx_2, sx_3) (-1)^{|x_{\sigma(1)}| + |x_{\sigma(2)}|} i_{x_{\sigma(3)} \wedge [x_{\sigma(2)}, x_{\sigma(1)}]} \omega \\ &= - \sum_{s \in Sh(2,1)} (-1)^\sigma e(s; sx_1, sx_2, sx_3) \{ \{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)} \} \\ & - \sum_{s \in Sh(1,2)} (-1)^\sigma e(s; sx_1, sx_2, sx_3) (-1)^{|x_{\sigma(1)}| + |x_{\sigma(2)}|} i_{[x_{\sigma(2)}, x_{\sigma(1)}]} df_{\sigma(3)} \\ &= - \sum_{s \in Sh(2,1)} (-1)^\sigma e(s; sx_1, sx_2, sx_3) \{ \{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)} \} \\ & - \sum_{s \in Sh(1,2)} (-1)^\sigma e(s; sx_1, sx_2, sx_3) \{ df_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)} \} \\ &= d\{f_1, f_2, f_3\}. \end{aligned}$$

□

*Remark.* At this point we should stress again, that if there is *no* tensor  $y$  which satisfies

$$i_y \omega = - \sum_{s \in Sh(2,1)} (-1)^\sigma e(s; sx_1, sx_2, sx_3) \{ \{f_{\sigma(1)}, f_{\sigma(2)}\}, f_{\sigma(3)} \},$$

then the previous proof shows, that the image  $\{f_1, f_2, f_3\}$  must not be Hamilton. Regarding example (6) this justifies our definition of Poisson cotensors as exterior cotensors satisfying both the fundamental equation and its Poisson constraint.

With the homotopy Poisson 3-bracket at hand, all higher Poisson brackets are now defined inductively in terms of contractions along Hamilton tensors of the previously defined bracket:



**Definition 3.17.** Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  the set of Poisson cotensors. The **homotopy Poisson  $k$ -bracket**

$$\{\cdot, \dots, \cdot\} : \times^k \mathcal{Pois}(A, \mathfrak{g}, \omega) \rightarrow \mathcal{Pois}(A, \mathfrak{g}, \omega) \quad (23)$$

is defined inductively for any  $k > 3$ , homogeneous  $f_1, \dots, f_k \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and Hamilton tensor  $x_{\{f_1, \dots, f_k\}}$  associated to the homotopy Poisson  $(k-1)$ -bracket by

$$\{f_1, \dots, f_k\} := (-1)^{k-1} \sum_{\sigma \in Sh(k-1, 1)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k-1)}\}}} f_{\sigma(k)}$$

and is then extended to all of  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  by linearity.

The induction base is the homotopy Poisson 3-bracket. If we refer to the Poisson  $k$ -bracket for arbitrary  $k \in \mathbb{N}$ , the differential  $d$  is usually meant to be the homotopy Poisson "1-bracket" and in this context sometimes written as  $\{\cdot\}$ .

The following theorem basically says, that the infinite sequence of homotopy Poisson  $k$ -brackets defines an  $(n-1)$ -fold shifted homotopy Lie algebra on Poisson cotensors.

**Theorem 3.18.** For any  $k \in \mathbb{N}$ , the homotopy Poisson  $k$ -bracket is well defined,  $(n-1)$ -fold shifted graded antisymmetric, homogeneous of degree  $(k-1)n-1$  and the  $(n-1)$ -fold shifted homotopy Jacobi equation

$$\sum_{j=1}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\} = 0$$

is satisfied for any homogeneous  $f_1, \dots, f_k \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$ . If  $y_1, \dots, y_k$  are Poisson constraints, associated to  $f_1, \dots, f_k$ , respectively, a Poisson constraint of their  $k$ -ary bracket is given by

$$y_{\{f_1, \dots, f_k\}} := - \sum_{\sigma \in Sh(k-1, 1)} (-1)^{\sigma+k} e(\sigma; y_1, \dots, y_k) y_{\sigma(k)} \wedge x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k-1)}\}} \quad (24)$$

If  $x_1, \dots, x_k$  are Hamilton tensors, associated to  $f_1, \dots, f_k$ , respectively, a Hamilton tensor of the  $k$ -ary bracket is given by

$$x_{\{f_1, \dots, f_k\}} := y_{J_k}(f_{\sigma(1)}, \dots, f_{\sigma(k)}) + \sum_{\sigma \in Sh(k-1, 1)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) (-1)^{\sum_{i=1}^{k-1} (|x_{\sigma(i)}|-1)} x_{\sigma(k)} \wedge x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k-1)}\}}, \quad (25)$$

where the 'higher Jacobi' tensor  $y_{J_k}(f_{\sigma(1)}, \dots, f_{\sigma(k)})$  is a Poisson constraint, defined by the equation

$$i_{y_{J_k}(f_{\sigma(1)}, \dots, f_{\sigma(k)})} \omega = - \sum_{j=2}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\} \quad (26)$$

*Proof.* (By induction) For  $k \leq 3$  this was shown previously. For the induction step assume that all statements of the theorem are true for some  $k \in \mathbb{N}$ . We proof that they are true for  $(k+1)$ :

First of all lets see that the definition does not depend on the particular chosen associated Hamilton tensor. This follows from proposition (3.5). Since the difference of tensors associated to the same Poisson cotensor differ only in elements of

the kernel of  $\omega$  we can find a  $\xi \in \ker(\omega)$  with  $i_{x'_{\{f_1, \dots, f_k\}}} f_{k+1} = i_{x_{\{f_1, \dots, f_k\}} + \xi} f_{k+1} = i_{x_{\{f_1, \dots, f_k\}}} f_{k+1}$  because each  $f_i$  has the kernel property.

To see the  $(n-1)$ -fold shifted graded antisymmetry, we use the assumed  $(n-1)$ -fold shifted graded antisymmetry of any associated Hamilton tensor  $x_{\{f_1, \dots, f_k\}}$  (up to elements of the kernel of  $\omega$ ) and rewrite the definition in terms of the symmetric group

$$-\frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+k} e(\sigma; sx_1, \dots, sx_k) i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k-1)}\}}} f_{\sigma(k)}.$$

This expression is  $(n-1)$ -fold shifted graded antisymmetric, since  $|sx_i| = |s^{n-1} f_i|$ .

To see the homogeneity, assume that every argument is homogeneous. Then

$$\begin{aligned} |\{f_1, \dots, f_k\}| &= |i_{x_{\{f_1, \dots, f_{k-1}\}}} f_k| = \\ &= |x_{\{f_1, \dots, f_{k-1}\}}| + |f_k| = \\ &= |x_1| + \dots + |x_{k-1}| + |f_k| - 1 = \\ &= |f_1| + \dots + |f_k| - 1 + (k-1)n. \end{aligned}$$

The proof of the  $(n-1)$ -fold shifted homotopy Jacobi equation is a very long calculations. According to a better readable text, we compute it in section (C).

To see that (24) is a Poisson constraint associated to  $\{f_1, \dots, f_k\}$  just compute the contraction  $i_{y_{\{f_1, \dots, f_k\}}} \omega$ .

To see that (25) is a Hamilton tensor associated to  $\{f_1, \dots, f_k\}$ , first observe that equation (26) always has a solution  $y_{J_k}(f_{\sigma(1)}, \dots, f_{\sigma(k)})$ , since the right side of the equation is a Poisson tensor by the induction hypothesis. Using this and the assumption that the homotopy Jacobi equation holds, we compute the contraction

$$\begin{aligned} & i_{x_{\{f_1, \dots, f_k\}}} \omega \\ &= i_{y_{J_k}(f_{\sigma(1)}, \dots, f_{\sigma(k)})} \omega + \sum_{\sigma \in Sh(k-1, 1)} (-1)^\sigma e(s; sx_1, \dots, sx_k) \\ & \quad (-1)^{\sum_{i=1}^{k-1} (|x_{\sigma(i)}| - 1)} i_{x_{\sigma(k)} \wedge x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k-1)}\}}} \omega \\ &= - \sum_{j=2}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \\ & \quad \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\} \\ & \quad + \sum_{\sigma \in Sh(k-1, 1)} (-1)^\sigma e(s; sx_1, \dots, sx_k) (-1)^{\sum_{i=1}^{k-1} (|x_{\sigma(i)}| - 1)} i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k-1)}\}}} df_{\sigma(k)} \\ &= - \sum_{j=2}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \\ & \quad \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\} \\ & \quad + \sum_{\sigma \in Sh(k-1, 1)} (-1)^\sigma e(s; sx_1, \dots, sx_k) (-1)^{\sum_{i=1}^{k-1} (|x_{\sigma(i)}| - 1)} \{f_{\sigma(1)}, \dots, f_{\sigma(k-1)}, df_{\sigma(k)}\} \\ &= - \sum_{j=2}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \\ & \quad \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\} \\ & \quad - \sum_{\sigma \in Sh(1, k-1)} (-1)^{\sigma+(k-1)} e(\sigma; sx_1, \dots, sx_k) \{df_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(k)}\} \\ &= - \sum_{j=1}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \\ & \quad \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\} \\ &= d\{f_1, \dots, f_k\}. \end{aligned}$$

□

**Corollary 3.19** (The shifted homotopy Lie algebra). *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  the graded linear space of Poisson cotensors. The sequence  $(\{\cdot, \dots, \cdot\})_{k \in \mathbb{N}}$  of all  $k$ -ary brackets then defines an  $(n-1)$ -fold shifted homotopy Lie algebra on  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$ .*

**3.6. The Leibniz operators.** In this section we look in detail at the Poisson-Leibniz equation. For  $n > 1$ , it does not hold strictly, but only up to certain correction terms, which are controlled by what we call the first  $n$ -plectic Leibniz operator. It behaves somewhat in between a product and a bracket.

This operator is enough to build the homotopy Poisson- $n$  structure in dimension two, but beyond that, the question remains how the higher Poisson brackets interact with the commutative structure. As it turns out, this is controlled in a similar manner, by higher Leibniz-like equations.

Again these equations don't hold strictly, but up to certain correction terms, which are controlled by additional operators. We call them the higher  $n$ -plectic Leibniz operators. They complete the homotopy Poisson- $n$  structure.

We start with the first Leibniz operator and look at all the Leibniz-equations in dimension two. After that, we define the general pattern for the higher Leibniz operators and derive the additional Leibniz-like equations for arbitrary dimension.

**Definition 3.20.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  the set of Poisson cotensors. The **first  $n$ -plectic Leibniz operator** is the map*

$$\{\cdot \parallel \cdot, \cdot\} : \times^3 \mathcal{Pois}(A, \mathfrak{g}, \omega) \rightarrow \mathcal{Pois}(A, \mathfrak{g}, \omega), \quad (27)$$

which is defined for any homogeneous  $f_1, f_2$  and  $f_3 \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and associated Hamilton tensors  $x_1, x_2$  and  $x_3$  by the equation

$$\begin{aligned} \{f_1 \parallel f_2, f_3\} = & -i_{x_1}(f_2 \wedge f_3) + (-1)^{(|x_1|-1)(|x_{f_2 \wedge f_3}|-1)} i_{x_{f_2 \wedge f_3}} f_1 \\ & + \left( i_{x_1} f_2 - (-1)^{(|x_1|-1)(|x_2|-1)} i_{x_2} f_1 \right) \wedge f_3 \\ & + (-1)^{|x_1||f_2|} f_2 \wedge \left( i_{x_1} f_3 - (-1)^{(|x_1|-1)(|x_3|-1)} i_{x_3} f_1 \right) \end{aligned} \quad (28)$$

and is then extended to all of  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  by linearity.

As the following theorem shows, this map is well defined, does not depend on the actual choice of any Hamilton tensor and interacts with the Poisson 2-bracket in terms of a homotopy Leibniz equation.

**Theorem 3.21.** *The first homotopy Leibniz operator  $\{\cdot \parallel \cdot, \cdot\}$  is well defined, trilinear, homogeneous of degree  $n$  with respect to the tensor grading and it satisfies the symmetry equation*

$$\{f_1 \parallel f_2, f_3\} = (-1)^{|f_2||f_3|} \{f_1 \parallel f_3, f_2\}. \quad (29)$$

for all (homogeneous)  $f_1, f_2$  and  $f_3$ . If  $x_1, x_2$  and  $x_3$  are associated Hamilton tensors, the following **first  $n$ -plectic Leibniz equation** in dimension two holds:

$$\begin{aligned} \{f_1, f_2 \wedge f_3\} = & \{f_1, f_2\} \wedge f_3 + (-1)^{(|x_1|-1)|f_2|} f_2 \wedge \{f_1, f_3\} \\ & + d\{f_1 \parallel f_2, f_3\} + \{df_1 \parallel f_2, f_3\} - (-1)^{|x_1|} \{f_1 \parallel df_2, f_3\} \\ & - (-1)^{|f_2|+|x_1|} \{f_1 \parallel f_2, df_3\}. \end{aligned} \quad (30)$$

If  $y_1, y_2$  resp.  $y_3$  are tensors associated to  $f_1, f_2$  resp.  $f_3$  by the equations  $i_{y_i}\omega = f_i$ , the exterior tensor

$$\begin{aligned} y_{\{f_1\|f_2, f_3\}} = & -j_{f_2}y_3 \wedge x_1 + (-1)^{(|x_1|-1)(|x_{f_2 \wedge f_3}|-1)}y_1 \wedge x_{f_2 \wedge f_3} \\ & + (-1)^{|x_1||f_2|+|f_2|(|x_1|+|f_3|)}j_{(i_{x_1}f_3 - (-1)^{(|x_1|-1)(|x_3|-1)}i_{x_3}f_1)}y_2 \\ & + j_{(i_{x_1}f_2 - (-1)^{(|x_1|-1)(|x_2|-1)}i_{x_2}f_1)}y_3 \end{aligned} \quad (31)$$

is Poisson constraint, associated to the first  $n$ -plectic Leibniz operator by the equation  $i_{y_{\{f_1\|f_2, f_3\}}}\omega = \{f_1\|f_2, f_3\}$ . Moreover the exterior tensor

$$\begin{aligned} x_{\{f_1\|f_2, f_3\}} := & y_{\{f_1, f_2 \wedge f_3\}} - y_{\{f_1, f_2\}} \wedge f_3 - (-1)^{(|x_1|-1)|f_2|}y_{f_2 \wedge \{f_1, f_3\}} \\ & - y_{\{df_1\|f_2, f_3\}} + (-1)^{|x_1|}y_{\{f_1\|df_2, f_3\}} + (-1)^{|f_2|+|x_1|}y_{\{f_1\|f_2, df_3\}} \end{aligned} \quad (32)$$

is a Hamilton tensor associated to the first  $n$ -plectic Leibniz operator by the equation  $i_{x_{\{f_1\|f_2, f_3\}}}\omega = d\{f_1\|f_2, f_3\}$ .

*Proof.* Trilinearity is immediate. To see that the definition does not depend on the particular chosen associated Hamilton tensors, we use (3.5) and proceed as in the proof of (3.11). To compute the homogeneity, recall  $|x| = |f| + n$ . Then

$$\begin{aligned} |\{f_1\|f_2, f_3\}| &= |i_{x_1}(f_2 \wedge f_3)| \\ &= |x_1| + |f_2| + |f_3| \\ &= |f_1| + |f_2| + |f_3| + n. \end{aligned}$$

The symmetry is a consequence of the graded symmetry of the exterior cotensor product.

To see the  $n$ -plectic Leibniz equation we compute the 'strict part' and the part that involves all the first  $n$ -plectic Leibniz operator separately. For the strict part we get

$$\begin{aligned} -\{f_1, f_2 \wedge f_3\} + \{f_1, f_2\} \wedge f_3 + (-1)^{(|x_1|-1)|f_2|}f_2 \wedge \{f_1, f_3\} = \\ + L_{x_1}(f_2 \wedge f_3) - (-1)^{(|x_1|-1)(|x_{f_2 \wedge f_3}|-1)}L_{x_{f_2 \wedge f_3}}f_1 \\ - (L_{x_1}f_2 - (-1)^{(|x_1|-1)(|x_2|-1)}L_{x_2}f_1) \wedge f_3 \\ - (-1)^{(|x_1|-1)|f_2|}f_2 \wedge (L_{x_1}f_3 - (-1)^{(|x_1|-1)(|x_3|-1)}L_{x_3}f_1). \end{aligned}$$

To compute the other part recall  $|x_{df}| = |x_f| - 1$ . After a long and tedious, but completely basic computation we get

$$\begin{aligned} d\{f_1\|f_2, f_3\} + \{df_1\|f_2, f_3\} - (-1)^{|x_1|}\{f_1\|df_2, f_3\} - (-1)^{|f_2|+|x_1|}\{f_1\|f_2, df_3\} = \\ - L_{x_1}(f_2 \wedge f_3) + (-1)^{(|x_1|+1)(|x_{f_2 \wedge f_3}|-1)}L_{x_{f_2 \wedge f_3}}f_1 \\ + (L_{x_1}f_2 - (-1)^{(|x_1|+1)(|x_2|+1)}L_{x_2}f_1) \wedge f_3 \\ + (-1)^{(|x_1|-1)|f_2|}f_2 \wedge (L_{x_1}f_3 - (-1)^{(|x_1|+1)(|x_3|+1)}L_{x_3}f_1) \\ - (-1)^{|x_1|+|x_1|+1)(|x_{df_2 \wedge f_3}|-1)}i_{x_{df_2 \wedge f_3}}f_1 \\ - (-1)^{|f_2|+|x_1|+1)(|x_{f_2 \wedge df_3}|-1)}i_{x_{f_2 \wedge df_3}}f_1. \end{aligned}$$

Since  $|x_{df_2 \wedge f_3}| = |x_{f_2 \wedge df_3}|$ , it follows that the homotopy Leibniz equation holds if the term

$$-(-1)^{|x_1|+1)(|x_{df_2 \wedge f_3}|-1)}i_{x_{df_2 \wedge f_3}}f_1 + (-1)^{|x_1|+1)(|x_{f_2 \wedge df_3}|-1)}i_{x_{f_2 \wedge df_3}}f_1$$

vanishes. This, however, is true since  $x_{df_2 \wedge f_3} + (-1)^{|v_2|} x_{f_2 \wedge df_3} = x_{d(f_2 \wedge f_3)} \in \ker(\omega)$  and  $f_1$  has the kernel property (3.5).

To show that the image  $\{f_1, \|f_2, f_2\}$  is a Poisson cotensors, it is enough to prove (31) and (32). To see (31) we use (10) and compute

$$\begin{aligned} i_{y_{\{f_1 \| f_2, f_3\}}} \omega &= -i_{j_{f_2} y_3 \wedge x_1} \omega + (-1)^{(|x_1|-1)(|x_{f_2 \wedge f_3}|-1)} i_{y_1 \wedge x_{f_2 \wedge f_3}} \omega \\ &\quad + (-1)^{|x_1| \| f_2 \| + |f_2| (|x_1| + |f_3|)} i_{j_{(i_{x_1} f_3 - (-1)^{(|x_1|-1)(|x_3|-1)} i_{x_3} f_1)} y_2} \omega \\ &\quad + i_{j_{(i_{x_1} f_2 - (-1)^{(|x_1|-1)(|x_2|-1)} i_{x_2} f_1)} y_3} \omega = \\ &= -i_{x_1} i_{j_{f_2} y_3} \omega + (-1)^{(|x_1|-1)(|x_{f_2 \wedge f_3}|-1)} i_{x_{f_2 \wedge f_3}} i_{y_1} \omega \\ &\quad + (-1)^{|x_1| \| f_2 \| + |f_2| (|x_1| + |f_3|)} (i_{x_1} f_3 - (-1)^{(|x_1|-1)(|x_3|-1)} i_{x_3} f_1) \wedge f_2 \\ &\quad + i_{j_{(i_{x_1} f_2 - (-1)^{(|x_1|-1)(|x_2|-1)} i_{x_2} f_1)} y_3} \omega = \{f_1 \| f_2, f_3\}. \end{aligned}$$

To see (32), observe that it is an immediate consequence of the  $n$ -plectic Leibniz equation, since all involved terms are Poisson cotensors. To be more precise, we compute

$$\begin{aligned} d\{f_1 \| f_2, f_3\} &= \\ &+ \{f_1, f_2 \wedge f_3\} - \{f_1, f_2\} \wedge f_3 - (-1)^{(|x_1|-1)|f_2|} f_2 \wedge \{f_1, f_3\} \\ &- \{df_1 \| f_2, f_3\} + (-1)^{|x_1|} \{f_1 \| df_2, f_3\} + (-1)^{|f_2| + |x_1|} \{f_1 \| f_2, df_3\} = \\ &= i_{y_{\{f_1, f_2 \wedge f_3\}}} \omega - i_{y_{\{f_1, f_2\} \wedge f_3}} \omega - (-1)^{(|x_1|-1)|f_2|} i_{y_{f_2 \wedge \{f_1, f_3\}}} \omega \\ &- i_{y_{\{df_1 \| f_2, f_3\}}} \omega + (-1)^{|x_1|} i_{y_{\{f_1 \| df_2, f_3\}}} \omega + (-1)^{|f_2| + |x_1|} i_{y_{\{f_1 \| f_2, df_3\}}} \omega \end{aligned}$$

□

*Remark.* We say the first  $n$ -plectic Leibniz equation is satisfied *strictly* or 'on the nose', if all terms which contain a Leibniz operator vanish. In that case the equation is precisely equal to the common Leibniz equation of a Poisson algebra. For  $n = 1$ , this is always the case.

Loosely speaking, the Leibniz equation controls the interaction between the Poisson bracket and the differential graded commutative structure. As we will see in corollary (3.23), this is exactly the general structure equation (49) for the parameters  $k = 2$ ,  $p_1 = 1$  and  $p_2 = 2$ .

However, two more Leibniz-like equations have to hold in case  $k = 2$ . They derive from (49), for the parameters  $p_1 = 1$  and  $p_2 = 3$  as well as  $p_1 = 2$  and  $p_2 = 2$ . The following theorem makes this precise:

**Theorem 3.22.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure. Then for any four Poisson cotensors  $f_1, \dots, f_4 \in \text{Pois}(A, \mathfrak{g}, \omega)$  and associated Hamilton tensors  $x_1, \dots, x_4$ , the following **second  $n$ -plectic Leibniz equation** in dimension two holds:*

$$\begin{aligned} \{f_1 \wedge f_2 \| f_3, f_4\} &- (-1)^{|x_{f_1 \wedge f_2}| |x_{f_3 \wedge f_4}|} \{f_3 \wedge f_4 \| f_1, f_2\} = \\ &= f_1 \wedge \{f_2 \| f_3, f_4\} + (-1)^{|f_2| |x_{f_3 \wedge f_4}|} \{f_1 \| f_3, f_4\} \wedge f_2 \\ &- (-1)^{|x_{f_1 \wedge f_2}| |x_{f_3 \wedge f_4}|} f_3 \wedge \{f_4 \| f_1, f_2\} \\ &- (-1)^{|x_{f_1 \wedge f_2}| (|x_{f_3 \wedge f_4}| + |f_4|)} \{f_3 \| f_1, f_2\} \wedge f_4. \quad (33) \end{aligned}$$

In addition the **third  $n$ -plectic Leibniz equation** in dimension two holds for the same arguments:

$$\begin{aligned} (-1)^{|f_2||x_1|} f_2 \wedge \{f_1 \| f_3, f_4\} - \{f_1 \| f_2, f_3\} \wedge f_4 = \\ - \{f_1 \| f_2, f_3 \wedge f_4\} + \{f_1 \| f_2 \wedge f_3, f_4\} \end{aligned} \quad (34)$$

*Proof.* To see that both equations are satisfied, it is enough to just apply the definition (28) of the first Leibniz operator to all expressions and collect the terms. This is a rather long, but simple computation and it is left to the reader.  $\square$

This is everything that needs to happen in 'dimension two', i.e. for the parameter  $k = 2$  in the defining structure equation (49). As the following corollary make precise, we can choose all structure maps (48) beside the exterior product, the Poisson bracket and the first Leibniz operator to be zero, in this case.

**Corollary 3.23.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure. The main structure equation (49) of a homotopy Poisson- $n$  algebra is satisfied for  $k = 2$  and all  $p_1, p_2 \in \mathbb{N}$ .*

*Proof.* Recall the structure in dimension one, as given in the proof of corollary (3.9) and define the following additional maps of (48) for  $k = 2$ : The Poisson-2 bracket is shifted into  $D_{1,1}(s^{n-1}f_1 \wedge s^{n-1}f_2) := s\{f_1, f_2\}$  and the first Leibniz operator is shifted into  $D_{1,2}(s^{n-1}f_1 \wedge s^{n-2}(\overline{sf_2 \otimes sf_3})) := (-1)^{|sf_2|} s\{f_1 \| f_2, f_3\}$ . In addition consider the permutation of the shifted Leibniz operator

$$\begin{aligned} D_{2,1}(s^{n-2}(\overline{sf_2 \otimes sf_3}) \wedge s^{n-1}f_1) := \\ - (-1)^{|s^{n-1}f_1||s^{n-2}(\overline{sf_2 \otimes sf_3})|} D_{1,2}(s^{n-1}f_1 \wedge s^{n-2}(\overline{sf_2 \otimes sf_3})) . \end{aligned}$$

All other structure maps  $D_{q_1, q_2}$  are assumed to be zero.

These maps are homogeneous of degree  $(2 - n)$  and have the expected symmetry. Since most maps are just zero, equation (49) is only non trivial in the following four cases:  $(p_1, p_2) \in \{(1, 1), (1, 2), (1, 3), (2, 2)\}$ .

The case  $(1, 1)$  is the requirement, that the de Rham differential is a derivation with respect to bracket. The cases  $(1, 2)$ ,  $(2, 2)$  and  $(1, 3)$  are the first, second and third  $n$ -plectic Leibniz equation, respectively.  $\square$

Finally, lets look at the higher Leibniz equations. Basically, they control the interaction between the higher Poisson brackets and the commutative structure. However, for them to hold we need additional structure: The higher Leibniz operators.

With the first Leibniz operator at hand, these operators are defined inductively. Unlike the higher brackets, contractions along Hamilton tensors of the previously defined bracket, is not enough. The following definition makes this precise:

**Definition 3.24.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure and  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  the set of Poisson cotensors. The  $k$ -th  **$n$ -plectic Leibniz operator** is the map*

$$\{\cdot \| \cdot, \cdot\} : \times^{k+2} \mathcal{Pois}(A, \mathfrak{g}, \omega) \rightarrow \mathcal{Pois}(A, \mathfrak{g}, \omega) , \quad (35)$$

*defined inductively for any  $k > 1$ , homogeneous  $f_1, \dots, f_{k+2} \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and Hamilton tensor  $x_{\{\cdot, \dots, \cdot, \|, \cdot\}}$  associated to the  $(k - 1)$ -th Leibniz operator by*

$$\begin{aligned}
\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\} := & \quad (36) \\
& - \sum_{\sigma \in Sh(k-1,1)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) (-1)^{k+(|x_{v_{k+1}} \wedge v_{k+2}| - 1)(|x_{\sigma(k)}| - 1)} \\
& \quad i_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-1)} \| v_{k+1}, v_{k+2}\}}} v_{\sigma(k)} \\
& + \sum_{j_1=0}^k \sum_{j_2=j_1}^k \sum_{\sigma \in Sh(j_1, j_2-j_1, k-j_2)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^{j_1 + \sum_{i=1}^k (k-i)(|x_{\sigma(i)}| - 1) + (\sum_{i=j_2+1}^k |x_{\sigma(i)}|) |v_{k+1}|} \\
& \quad i_{x_{\sigma(1)}} \dots i_{x_{\sigma(j_1)}} ((i_{x_{\sigma(j_1+1)}} \dots i_{x_{\sigma(j_2)}} v_{k+1}) \wedge (i_{x_{\sigma(j_2+1)}} \dots i_{x_{\sigma(k)}} v_{k+2})) \\
& - \sum_{j_1=0}^{k-1} \sum_{j_2=j_1}^{k-1} \sum_{\sigma \in Sh(j_1, j_2-j_1, 1, k-1-j_2)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^{j_1 + \sum_{i=1}^k (k-i)(|x_{\sigma(i)}| - 1) + (\sum_{i=j_2+2}^k |x_{\sigma(i)}|) |v_{k+1}| + (|x_{\sigma(j_2+1)}| + 1)(|x_{k+1}| + 1)} \\
& \quad i_{x_{\sigma(1)}} \dots i_{x_{\sigma(j_1)}} ((i_{x_{\sigma(j_1+1)}} \dots i_{x_{\sigma(j_2)}} i_{x_{k+1}} v_{\sigma(j_2+1)}) \wedge (i_{x_{\sigma(j_2+2)}} \dots i_{x_{\sigma(k)}} v_{k+2})) \\
& - \sum_{j_1=0}^{k-1} \sum_{j_2=j_1}^{k-1} \sum_{\sigma \in Sh(j_1, j_2-j_1, k-1-j_2, 1)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^{j_1 + \sum_{i=1}^k (k-i)(|x_{\sigma(i)}| - 1) + (\sum_{i=j_2+1}^k |x_{\sigma(i)}|) |v_{k+1}| + (|x_{\sigma(k)}| + 1)(|x_{k+2}| + 1)} \\
& \quad i_{x_{\sigma(1)}} \dots i_{x_{\sigma(j_1)}} ((i_{x_{\sigma(j_1+1)}} \dots i_{x_{\sigma(j_2)}} v_{k+1}) \wedge (i_{x_{\sigma(j_2+1)}} \dots i_{x_{\sigma(k-1)}} i_{x_{k+2}} v_{\sigma(k)})) \\
& + \sum_{j_1=0}^{k-2} \sum_{j_2=j_1}^{k-2} \sum_{\sigma \in Sh(j_1, j_2-j_1, 1, k-2-j_2, 1)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^{j_1 + \sum_{i=1}^k (k-i)(|x_{\sigma(i)}| - 1) + (\sum_{i=j_2+2}^k |x_{\sigma(i)}|) |v_{k+1}|} \\
& \quad (-1)^{(|x_{\sigma(j_2+1)}| + 1)(|x_{k+1}| + 1) + (|x_{\sigma(k)}| + 1)(|x_{k+2}| + 1)} \\
& \quad i_{x_{\sigma(1)}} \dots i_{x_{\sigma(j_1)}} ((i_{x_{\sigma(j_1+1)}} \dots i_{x_{\sigma(j_2)}} i_{x_{k+1}} v_{\sigma(j_2+1)}) \wedge ( \\
& \quad \quad i_{x_{\sigma(j_2+2)}} \dots i_{x_{\sigma(k-1)}} i_{x_{k+2}} v_{\sigma(k)}))
\end{aligned}$$

and is then extended to all of  $\mathcal{Pois}(A, \mathfrak{g}, \omega)$  by linearity.

The induction base is the first Leibniz operator. The idea of the notation  $\{\cdot, \dots, \cdot \| \cdot, \cdot\}$  is, that everything on the left of  $\|$  'is like a bracket', while everything on the right 'is like a product'. This is meant in particular with respect to homogeneity and symmetry.

*Remark.* In actual computations it is sometimes advantageous to consider the commutative product as the 'zeroth-Leibniz operator'. In that case we follow the notational logic of the higher Leibniz operators and write  $\{\|f_1, f_2\}$  for the product  $f_1 \wedge f_2$ .

The following theorem basically says, that the infinite sequence of higher Leibniz operators is well defined and interact with the commutative structure and the Poisson brackets in terms of a homotopy Poisson- $n$  algebra:

**Theorem 3.25.** *For any  $k \in \mathbb{N}$ , the  $k$ -th Leibniz operator is well defined,  $(n-1)$ -fold shifted graded antisymmetric with respect to all arguments on the left and graded symmetric with respect to all arguments on the right. Moreover it is homogeneous of degree  $n \cdot k$  with respect to the tensor grading.*

For any homogeneous Poisson cotensors  $f_1, \dots, f_{k+2} \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and associated (homogeneous) Hamilton tensor  $x_1, \dots, x_{k+2}$ , the following **first  $n$ -plectic Leibniz equation** in dimension  $(k+1)$  is satisfied:

$$\begin{aligned}
& -\{f_1, \dots, f_k, f_{k+1} \wedge f_{k+2}\} + \{f_1, \dots, f_k, f_{k+1}\} \wedge f_{k+2} \\
& \quad + (-1)^{(\sum_{i=1}^k |x_i| - 1)|f_{k+1}|} f_{k+1} \wedge \{f_1, \dots, f_k, f_{k+2}\} \\
& + \sum_{\sigma \in Sh(1, k-1)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot \{df_{\sigma(1)}, f_{\sigma(2)}, \dots, f_{\sigma(k)} \| f_{k+1}, f_{k+2}\} \\
& \quad - (-1)^k d\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\} \\
& + (-1)^{\sum_{i=1}^k (|x_i| - 1)} \{f_1, \dots, f_k \| df_{k+1}, f_{k+2}\} \\
& \quad + (-1)^{\sum_{i=1}^k (|x_i| - 1) + |f_{k+1}|} \{f_1, \dots, f_k \| f_{k+1}, df_{k+2}\} \\
& + \sum_{j=2}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) (-1)^{(j+1)(k+1-j)} \cdot \\
& \quad \cdot \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)} \| f_{k+1}, f_{k+2}\} \\
& - \sum_{j=1}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot (-1)^{j(k-j)+k+(|x_{f_{k+1} \wedge f_{k+2}}| - 1)(\sum_{i=j+1}^k (|x_{\sigma(i)}| - 1))} \cdot \\
& \quad \cdot \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)} \| f_{k+1}, f_{k+2}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\} \\
& + \sum_{j=1}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot (-1)^{(\sum_{i=k-j+1}^k |x_{\sigma(i)}| - 1)|f_{k+1}| + (j-1)\sum_{i=1}^{k-j} (|x_{\sigma(i)}| - 1)} \cdot \\
& \quad \cdot \{f_{\sigma(1)}, \dots, f_{\sigma(k-j)} \| f_{k+1}, \{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}, f_{k+2}\}\} \\
& + \sum_{j=1}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) (-1)^{(j-1)(\sum_{i=1}^{k-j} (|x_{\sigma(i)}| - 1))} \cdot \\
& \quad \cdot \{f_{\sigma(1)}, \dots, f_{\sigma(k-j)} \| \{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}, f_{k+1}\}, f_{k+2}\} \\
& = 0.
\end{aligned} \tag{37}$$

*Proof.* Multi-linearity is immediate. To see that the definition does not depend on the particular chosen associated Hamilton tensors, we use (3.5) and proceed as in the proof of (3.11). To compute the homogeneity, recall  $|x| = |f| + n$ . Then

$$\begin{aligned}
|\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}| &= |i_{x_1 \wedge \dots \wedge x_k} (f_{k+1} \wedge f_{k+2})| \\
&= \sum_{i=1}^k |x_i| + |f_{k+1}| + |f_{k+2}| \\
&= \sum_{i=1}^k |f_i| + |f_2| + |f_3| + kn.
\end{aligned}$$

The graded symmetry for arguments on the right is a consequence of the graded symmetry of the exterior cotensor product. The  $(n-1)$ -fold shifted graded anti-symmetry for the arguments on the left, is a consequence of the degree identity  $|s^{n-1} f_i| = (|x_i| - 1)$  and the decalage sign  $(-1)^{\sum_{i=1}^k (|x_i| - 1)}$  which appears in the definition.

To see that  $\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}$  is a Poisson cotensor, for all Poisson cotensors  $f_1, \dots, f_{k+2}$ , we have to show that both equations

$$\begin{aligned}
i_{x_{\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}}} &= d\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\} \\
i_{y_{\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}}} &= \{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}
\end{aligned}$$

have tensor solutions. We proof this by induction on  $k$ . From the proof of theorem (3.21), we know that both equations are satisfied for  $k = 1$  and that the first  $n$ -plectic Leibniz equation holds for  $k = 1$ , too. This serves as the induction base.



For the induction step, suppose that there is a  $k \in \mathbb{N}$ , such that we already know that  $\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}$  is a Poisson cotensor and that the first  $n$ -plectic Leibniz equation holds for all  $j \leq k$ .

Then we need to show, that the first  $n$ -plectic Leibniz equation is moreover satisfied for  $k+1$ . Unfortunately the present proof of this equation is extremely long and therefore we only sketch the proof for now. What we do is basically just to apply the definition of the various Leibniz operators and the Poisson brackets to each equation and then collect terms. What remains is a contraction along a tensor, which is the Hamilton tensor of the same equation but for  $(k-1)$ . By the induction hypothesis, this tensor is an element of the kernel of  $\omega$  and therefore of any Poisson tensor by the kernel property (3.5). This completes the computation.

Now since the  $n$ -plectic Leibniz equation is satisfied for all  $j \leq k+1$ , each term is in particular a Poisson cotensor and therefore a solution to the equation  $i_{y_{\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}}} = \{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}$  exists.

A solution to  $i_{x_{\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}}} = d\{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\}$  then follows from the  $n$ -plectic Leibniz equation in dimension  $k+1$ , too.  $\square$

*Remark.* We say that the first  $n$ -plectic Leibniz equation holds *strictly* or 'on the nose', if all terms which involve Leibniz operators vanish. In that case we could say that the  $n$ -plectic Poisson  $k$ -bracket acts as some kind of derivation with respect to the exterior product.

Loosely speaking we can say, that the first higher Leibniz equations, control the interaction between the various Poisson brackets and the differential graded commutative structure. As we will see in corollary (3.27), this is precisely the general structure equation (49) for the parameters  $k \in \mathbb{N}$  and  $p_1 = 1, \dots, p_k = 1$  and  $p_{k+1} = 2$ .

However, two more Leibniz-like equations have to hold in any dimension  $k$ . They derive from (49) for the parameters  $p_1 = 1, \dots, p_k = 1, p_{k+1} = 3$  as well as  $p_1 = 1, \dots, p_k = 1, p_{k+1} = 2, p_{k+2} = 2$ . The following theorem makes this precise:

**Theorem 3.26.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure. Then for any Poisson cotensors  $f_1, \dots, f_{k+4} \in \text{Pois}(A, \mathfrak{g}, \omega)$  and associated Hamilton tensors  $x_1, \dots, x_{k+4}$ , the following **second  $n$ -plectic Leibniz equation** in dimension  $(k+2)$  holds:*

$$\begin{aligned}
& - \sum_{j=0}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k+1-j)} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^{|x_{f_{k+1}} \wedge f_{k+2}| \sum_{i=j+1}^k (|x_{\sigma(i)}| - 1)} \\
& \quad \{ \{ f_{\sigma(1)}, \dots, f_{\sigma(j)} \| f_{k+1}, f_{k+2} \}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)} \| f_{k+3}, f_{k+4} \} \\
& + \sum_{j=0}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k+1-j)} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^{\sum_{i=j+1}^k (|x_{\sigma(i)}| - 1) + |x_{f_{k+1}} \wedge f_{k+2}| |x_{f_{k+3}} \wedge f_{k+4}|} \\
& \quad \{ \{ f_{\sigma(1)}, \dots, f_{\sigma(j)} \| f_{k+3}, f_{k+4} \}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)} \| f_{k+1}, f_{k+2} \} \\
& + \sum_{j=0}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+k} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^{j(\sum_{i=1}^{k-j} (|x_{\sigma(i)}| - 1) + |f_{k+1}|) + (\sum_{i=k-j+1}^k (|x_{\sigma(i)}| - 1))(|f_{k+1}| + 1)} \\
& \quad \{ f_{\sigma(1)}, \dots, f_{\sigma(k-j)} \| f_{k+1}, \{ f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)} \| f_{k+2} \| f_{k+3}, f_{k+4} \} \}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+(k-j)} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^j \sum_{i=1}^{k-j} (|x_{\sigma(i)}| - 1) + |f_{k+2}| |x_{f_{k+3}} \wedge f_{k+4}| + \sum_{i=k-j+1}^k |x_{\sigma(i)}| \\
& \quad \{f_{\sigma(1)}, \dots, f_{\sigma(k-j)}\} \|\{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}, f_{k+1}\} \| f_{k+3}, f_{k+4}\}, f_{k+2}\} \\
& - \sum_{j=0}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+(k-j)} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^j \sum_{i=1}^{k-j} (|x_{\sigma(i)}| - 1) + |x_{f_{k+1}} \wedge f_{k+2}| |x_{k+3}| + \sum_{i=k-j+1}^k |x_{\sigma(i)}| \\
& \quad \{f_{\sigma(1)}, \dots, f_{\sigma(k-j)}\} \|\{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}, f_{k+3}\} \| f_{k+1}, f_{k+2}\}, f_{k+4}\} \\
& - \sum_{j=0}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+(k-j)} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^j |f_{k+3}| + j \sum_{i=1}^{k-j} (|x_{\sigma(i)}| - 1) + \sum_{i=k-j+1}^k |x_{\sigma(i)}| + |x_{f_{k+1}} \wedge f_{k+2}| |x_{f_{k+3}} \wedge f_{k+4}| \\
& \quad \{f_{\sigma(1)}, \dots, f_{\sigma(k-j)}\} \| f_{k+3}, \{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}, f_{k+4}\} \| f_{k+1}, f_{k+2}\} \\
& = 0. \tag{38}
\end{aligned}$$

In addition, for any Poisson cotensors  $f_1, \dots, f_{k+3} \in \mathcal{Pois}(A, \mathfrak{g}, \omega)$  and associated Hamilton tensors  $x_1, \dots, x_{k+3}$ , the **thirs  $n$ -plectic Leibniz equation** in dimension  $(k+2)$  holds:

$$0 = \{f_1, \dots, f_k \| f_{k+1} \wedge f_{k+2}, f_{k+3}\} \tag{39}$$

$$- \{f_1, \dots, f_k \| f_{k+1}, f_{k+2} \wedge f_{k+3}\} \tag{40}$$

$$+ \{f_1, \dots, f_k \| f_{k+1}, f_{k+2}\} \wedge f_{k+3} \tag{41}$$

$$- (-1)^{\sum_{i=1}^k |x_i|} |f_{k+1}| f_{k+1} \wedge \{f_1, \dots, f_k \| f_{k+2}, f_{k+3}\} \tag{42}$$

$$\begin{aligned}
& + \sum_{j=1}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j} \sum_{i=1}^{k-j} (|x_{\sigma(i)}| - 1) e(\sigma; sx_1, \dots, sx_k) \\
& \quad \{f_{\sigma(1)}, \dots, f_{\sigma(k-j)}\} \|\{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}\} \| f_{k+1}, f_{k+2}\}, f_{k+3}\} \\
& \tag{43}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^{k-1} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^j \sum_{i=1}^{k-j} (|x_{\sigma(i)}| - 1) + \sum_{i=k-j+1}^k |x_{\sigma(i)}| |f_{k+1}| \\
& \quad \{f_{\sigma(1)}, \dots, f_{\sigma(k-j)}\} \| f_{k+1}, \{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}\} \| f_{k+2}, f_{k+3}\} \\
& \tag{44}
\end{aligned}$$

*Proof.* The computation is extremely long and therefore we only sketch the proof for now. What we do is basically just to apply the definition of the various Leibniz operators to each equation and then just collect terms, using the associativity of the exterior product. What remains is a contraction along a tensor, which is the Hamilton tensor of the same equation but for  $(k-1)$ . By the induction hypothesis, this tensor is an element of the kernel of  $\omega$  and therefore of any Poisson tensor by the kernel property (3.5). This completes the computation.  $\square$

The following corollary summaries the partial steps for the various computations we did to derive the  $n$ -plectic homotopy Poisson- $n$  algebra of Poisson cotensors in higher symplectic geometry:

**Corollary 3.27.** *Let  $(A, \mathfrak{g}, \omega)$  be an  $n$ -plectic structure. The main structure equation (4.9) of a homotopy Poisson- $n$  algebra is satisfied for any  $k \in \mathbb{N}$  and all  $p_1, \dots, p_k \in \mathbb{N}$ .*

*Proof.* Recall the structure in dimension  $k \leq 2$ , as given in the proof of corollary (3.9) and (3.23). Define the following additional maps of (48) for  $k \geq 3$ :

The shifted brackets  $D_{1,\dots,1}(s^{n-1}f_1 \wedge \dots \wedge s^{n-1}f_k) := s\{f_1, \dots, f_k\}$ , as well as the shifted Leibniz operators  $D_{1,\dots,1,2}(s^{n-1}f_1 \wedge \dots \wedge s^{n-1}f_k \wedge s^{n-2}(sf_{k+1} \otimes sf_{k+2})) := (-1)^{|sf_{k+1}|}s\{f_1, \dots, f_k || f_{k+1}, f_{k+2}\}$ . the operators  $D_{1,\dots,1,2,1,\dots,1}$ , are defined, similar to the one in the proof of (3.23), by permuting the arguments in  $D_{1,\dots,1,2}$  accordingly. All other structure maps  $D_{q_1,\dots,q_k}$  are assumed to be zero.

These maps are homogeneous of degree  $(k-n)$  and have the expected symmetry. Since most maps are just zero, equation (49) is only non trivial in the following four cases:  $(p_1, \dots, p_k) \in \{(1, \dots, 1), (1, \dots, 1, 2), (1, \dots, 1, 3), (1, \dots, 1, 2, 2)\}$ .

The cases  $(1, \dots, 1)$  are precisely the  $(n-1)$ -fold shifted homotopy Jacoby equations. The cases  $(1, \dots, 1, 2)$ ,  $(1, \dots, 1, 2, 2)$  and  $(1, \dots, 1, 3)$  are the first, second and third  $n$ -plectic Leibniz equation in dimension  $k$ , respectively.  $\square$

## APPENDIX A. A PRIMER ON HOMOTOPY POISSON-N ALGEBRAS

This chapter aims to give a short 'generators and relations'-style introduction to homotopy Poisson- $n$  algebras. By definition these are algebras over the Koszul resolution  $\Omega p_n^!$  of the Poisson- $n$  operads  $p_n$ . For a more elaborate introduction see [6].

We start with an introduction to the very basic notations of shuffles and graded vector spaces. Then we look at the linear space that underlays the cofree Poisson- $n$  coalgebra. After that we make heavy use of the ideas from [5] to derive the explicit structure of a homotopy Poisson- $n$  algebra in terms of generating structure maps.

**A.1. Shuffle Permutation.** Let  $S_k$  be the symmetric group, i.e the group of all bijective maps of the ordinal  $[k]$ . Then for any  $p, q \in \mathbb{N}$  a  $(p, q)$ -shuffle is a permutation  $(\mu(1), \dots, \mu(p), \nu(1), \dots, \nu(q)) \in S_{p+q}$  subject to the condition  $\mu(1) < \dots < \mu(p)$  and  $\nu(1) < \dots < \nu(q)$ . We write  $Sh(p, q)$  for the set of all  $(p, q)$ -shuffles. For more on shuffles, see for example at [18].

*Remark.* Another common definition of a shuffle is a permutation subject to the condition  $\mu^{-1}(1) < \dots < \mu^{-1}(p)$  and  $\nu^{-1}(1) < \dots < \nu^{-1}(q)$ . However we stick to our definition above and call a permutation with the latter property an  $(p, q)$ -unshuffle.

**A.2. Graded Vector Spaces.** We recall the most basics facts about  $\mathbb{Z}$ -graded  $\mathbb{R}$ -vector spaces (just graded vector spaces for short).

A  $\mathbb{Z}$ -graded vector space  $V$  is the direct sum  $\bigoplus_{n \in \mathbb{Z}} V_n$  of vector spaces  $V_n$ . An element  $v \in V$  is said to be **homogeneous** of degree  $n$ , written as  $|x| = n$ , if it is in the image of the natural inclusion  $i_n : V_n \rightarrow V$ , which comes from the direct sum. Every vector has a decomposition into homogeneous elements.

A morphism  $f : V \rightarrow W$  of graded vector spaces, homogeneous of degree  $r$ , is a sequence of linear maps  $f_n : V_n \rightarrow W_{n+r}$  for all  $n \in \mathbb{Z}$ . The integer  $r$  is called the degree of  $f$  and is denoted by  $|f|$ .

A  $k$ -linear morphism  $f : V_1 \times \dots \times V_k \rightarrow W$  of graded vector spaces, homogeneous of degree  $r$ , is a sequence of  $k$ -linear maps  $f_{n_1, \dots, n_k} : (V_1)_{n_1} \times \dots \times (V_k)_{n_k} \rightarrow W_{\sum n_i + r}$  for all  $n_i \in \mathbb{Z}$ .

The (graded) tensor product  $V \otimes W$  of two graded vector spaces  $V$  and  $W$  is given by

$$(V \otimes W)_n := \bigoplus_{i+j=n} (V_i \otimes W_j)$$

and the twisting morphism is given by  $\tau : V \otimes W \rightarrow W \otimes V$  on homogeneous elements  $v \otimes w \in V \otimes W$  by

$$\tau(v \otimes w) := (-1)^{|v||w|} w \otimes v$$

and then extended to  $V \otimes W$  by linearity. The category of  $\mathbb{Z}$ -graded vector spaces is symmetric monoidal, with respect to this tensor product and twisting morphism.

**A.2.1. The Koszul sign.** Let  $V$  be a  $\mathbb{Z}$ -graded vector space,  $\sigma \in S_k$  a permutation and let  $v_1, \dots, v_k \in V$  be homogeneous vectors. Then the **Koszul sign**  $e(\sigma; v_1, \dots, v_k) \in \{-1, +1\}$  of the permutation and the vectors is defined by the equation

$$v_1 \otimes \dots \otimes v_k = e(\sigma; v_1, \dots, v_k) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}. \quad (45)$$

In an actual computation the Koszul sign can be computed by the following rules: If a permutation  $\sigma \in S_k$  is a transposition of consecutive neighbors only, i.e. it is a transposition  $j \leftrightarrow j+1$ , then  $e(\sigma; v_1, \dots, v_k) = (-1)^{|v_j||v_{j+1}|}$ .

If  $\tau \in S_k$  is another permutation, then the Koszul sign of the composition is the composition of the Koszul signs, i.e.

$$e(\tau\sigma; v_1, \dots, v_k) = e(\tau; v_{\sigma(1)}, \dots, v_{\sigma(k)}) e(\sigma; v_1, \dots, v_k).$$

Since any permutation can be realized as a composition of transpositions of consecutive neighbors, these rules are enough to compute any Koszul sign.

*Example 7.* Let  $V$  be a  $\mathbb{Z}$ -graded vector space,  $v_1, \dots, v_3 \in V$  be homogeneous vectors and  $\sigma \in S_3$  be given by  $(3, 1, 2)$ . Then the associated Koszul sign is given by

$$e(\sigma; v_1, v_2, v_3) = (-1)^{|v_1||v_3| + |v_2||v_3|},$$

since the permutation can be realized by two transpositions of consecutive neighbors and for each such transposition, we can apply the twisting morphism.

**A.3. Homotopy Poisson- $n$  algebras.** Most of this section is taken, more or less, from the work of Galvez-Cerrillo, Tonks and Vallette [5] on homotopy Gerstenhaber algebras. Only slight adoptions are necessary to deal with the situation of Poisson- $n$  algebras for arbitrary  $n$ .

If  $V$  is a differential graded module, then  $s^j V$  means the  $j$ -fold shifting of  $V$ . Moreover, given any vectors  $v_1, \dots, v_k \in sV$ , permutation  $\sigma \in S_k$  and interval  $[i, j] = \{i, i+1, \dots, j\}$ ,  $1 \leq i \leq j \leq k$ , we use the notation:

$$\begin{aligned} v_{[i,j]} &:= v_i \otimes v_{i+1} \otimes \dots \otimes v_j \\ v_{[i,j]}^\sigma &:= v_{\sigma^{-1}(i)} \otimes v_{\sigma^{-1}(i+1)} \otimes \dots \otimes v_{\sigma^{-1}(j)} \end{aligned}$$

**A.3.1. The cofree Poisson- $n$  coalgebra.** For any natural number  $n \in \mathbb{N}$ , let  $p_n$  be the **Poisson- $n$**  operad as for example introduced in Getzler and Jonas [6]. This operad is Koszul and can be seen as a composition of the commutative operad  $Com$  and the  $(n-1)$ -fold shifted Lie operad  $S^{n-1}\mathcal{L}ie$  by certain distributive laws.

$$p_n \simeq Com \circ S^{n-1}\mathcal{L}ie.$$

By the general algorithm [11], the structure equations of a *homotopy*  $p_n$ -algebra are encoded as a degree  $-1$  coderivation on the locally nilpotent, cofree  $p_n$ -coalgebra, which, for any differential graded (dg) module  $(V, d)$ , is given by

$$p_n^i(V) = s^{-1} p_n^c(sV) = s^{-n} Com^c(s^{n-1}\mathcal{L}ie^c(sV)).$$

For any (dg) module  $W$ ,  $\mathcal{Com}^c(W)$  is the locally nilpotent, cofree, commutative and coassociative coalgebra, which has the symmetric exterior tensor power  $SW := \bigoplus_{j \in \mathbb{N}} \odot^j W$  as underlying linear space. Moreover  $\mathcal{Lie}^c(sV)$  is the cofree Lie coalgebra. The underlying graded vector space is given by

$$\mathcal{Lie}^c(sV) = \bigoplus_{j \in \mathbb{N}} \overline{(sV)^{\otimes j}},$$

where  $\overline{(sV)^{\otimes k}}$  is the quotient of the tensor power  $(sV)^{\otimes k}$  by the images of the *shuffle maps*

$$Sh_{(i, k-i)} : (sV)^{\otimes i} \otimes (sV)^{\otimes (k-i)} \rightarrow (sV)^{\otimes k} \quad (46)$$

for all  $(1 \leq i \leq k-1)$ . These maps are defined for simple tensors  $(v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_k)$  by

$$\begin{aligned} Sh_{(i, k-i)}((v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_k)) = \\ \sum_{\sigma \in Sh_{(i, k-i)}} e(\sigma; v_1, \dots, v_k) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)} \end{aligned}$$

and are then extended to all of  $(sV)^{\otimes i} \otimes (sV)^{\otimes (k-i)}$  by linearity.

From this follows that the underlying linear space of the cofree  $p_n$ -coalgebra is given by

$$\begin{aligned} p_n^i(V) \simeq \bigoplus_{k \in \mathbb{N}} \bigoplus_{q_1 \leq \dots \leq q_k} s^{-n} \left( \left( s^{n-1} \overline{sV^{\otimes q_1}} \right) \odot \dots \odot \left( s^{n-1} \overline{sV^{\otimes q_k}} \right) \right) \simeq \\ \bigoplus_{k \in \mathbb{N}} \bigoplus_{q_1 \leq \dots \leq q_k} s^{k-n} \left( \left( s^{n-2} \overline{sV^{\otimes q_1}} \right) \wedge \dots \wedge \left( s^{n-2} \overline{sV^{\otimes q_k}} \right) \right) \end{aligned}$$

and that any linear map  $D : p_n^i(V) \rightarrow V$ , homogeneous of degree  $-1$  can equivalently be defined by a family of linear maps

$$D_{q_1, \dots, q_k} : \left( s^{n-2} \overline{sV^{\otimes q_1}} \right) \wedge \dots \wedge \left( s^{n-2} \overline{sV^{\otimes q_k}} \right) \rightarrow sV,$$

which are homogeneous of degree  $k-n$  and are indexed by  $k, q_1, \dots, q_k \in \mathbb{N}$ .

**A.3.2. The generating structure maps.** The following definition of a straight unshuffle is taken more or less verbatim from [5], where it is called a straight shuffle instead. The difference is due to a different approach to shuffles.

**Definition A.1** (Straight unshuffle). *Consider integers  $k \geq 1$  and  $l_j, r_j \geq 0$ ,  $q_j \geq 1$ ,  $p_k = l_k + q_k + r_k$  for each  $j = 1, \dots, k+1$ , and let*

$$P_j = \sum_{i=1}^{j-1} p_i, \quad L_j = \sum_{i=1}^k l_i + \sum_{i=1}^{j-1} q_i.$$

*Then a straight  $(l_j, q_j, r_j, p_j)_{j=1}^k$ -unshuffle is a  $(p_1, \dots, p_k)$ -unshuffle  $\sigma$  satisfying the following extra property for each  $j = 1, \dots, k$ :*

$$\sigma[P_j + l_j + 1, P_j + l_j + q_j] = [L_j + 1, L_{j+1}].$$

*By a straight  $(q_j, p_j)_{j=1}^k$ -unshuffle we mean a straight  $(l_j, q_j, r_j, p_j)_{j=1}^k$ -unshuffle for some values of  $l_j, r_j \geq 0$  with  $l_j + r_j = p_j - q_j$ .*

Less formally, a straight unshuffle may be thought of as a permutation  $\sigma$  of a concatenation  $X$  of  $k$  strings of lengths  $p_j$ , each of which contains a non-empty distinguished interval of length  $q_j$ , with  $l_j$  elements on the left and  $r_j$  elements on the right. For example,

$$X = 1 \underline{2} 3 4 \mid \underline{5} 6 7 8 \mid 9 \underline{10} \underline{11} \mid \underline{12} \underline{13}.$$

For the permutation to be a straight unshuffle it must satisfy following conditions:

- the orders of the elements within the  $k$  strings are preserved by  $\sigma$  (*unshuffle*),
- the distinguished intervals appear unchanged and contiguously in the image (*straight*).

For example, one possible straight unshuffles of  $X$  is

$$9 \ 1 \ \underline{2 \ 5 \ 6 \ 10 \ 11 \ 12 \ 13} \ 7 \ 3 \ 8 \ 4.$$

More precisely, it is a straight  $((1, 0, 1, 0), (1, 2, 2, 2), (2, 2, 0, 0), (4, 4, 3, 2))$ -unshuffles.

Using this definition, we can extend the straight shuffle extensions from [5] to the situation of Poisson- $n$  algebras for arbitrary  $n \in \mathbb{N}$ .

**Definition A.2** (Shifted straight shuffle extension). *Suppose we have natural numbers  $n, k, q_1, \dots, q_k \in \mathbb{N}$  and a graded linear map*

$$D_{q_1, \dots, q_k} : \left( s^{n-2} \overline{sV^{\otimes q_1}} \right) \wedge \dots \wedge \left( s^{n-2} \overline{sV^{\otimes q_k}} \right) \rightarrow sV,$$

*which is homogeneous of degree  $k - n$ . Then, for any integers  $p_k \geq q_k$ , the appropriate  $(n - 2)$ -fold **shifted straight shuffle extension** of  $D_{q_1, \dots, q_k}$  is the graded linear map*

$$D_{q_1, \dots, q_k}^{p_1, \dots, p_k} : \left( s^{n-2} \overline{sV^{\otimes p_1}} \right) \wedge \dots \wedge \left( s^{n-2} \overline{sV^{\otimes p_k}} \right) \rightarrow \left( s^{n-2} \overline{sV^{\otimes p}} \right) \quad (47)$$

*with  $p = 1 + \sum_{i=1}^k (p_i - q_i)$ , defined for any simple tensor  $\overline{v_{[1, p_1]}} \wedge \dots \wedge \overline{v_{[P_k+1, P_{k+1}]}} \in \left( s^{n-2} \overline{sV^{\otimes p_1}} \right) \wedge \dots \wedge \left( s^{n-2} \overline{sV^{\otimes p_k}} \right)$  by*

$$\sum_{\sigma} e(\sigma; v_1, \dots, v_{p_1+\dots+p_k}) (-1)^{k \cdot |v_{[1, L_1]}^{\sigma}|} \cdot s^{n-2} \left( \overline{v_{[1, L_1]}^{\sigma} \otimes D_{q_1, \dots, q_k} (s^{n-2} \overline{v_{[L_1+1, L_2]}^{\sigma}} \wedge \dots \wedge s^{n-2} \overline{v_{[L_k+1, L_{k+1}]}^{\sigma}}) \otimes v_{[L_{k+1}+1, P_{k+1}]}^{\sigma}} \right)$$

*Here  $\sigma$  runs over all straight  $(q_j, p_j)_{j=1}^k$ -unshuffle and the integers  $P_k, L_k$  are as above.*

Using these shifted straight shuffle extensions, we can define the structure of a homotopy Poisson- $n$  algebra in terms of generating functions and their relations. The following definition makes this precise:

**Definition A.3.** *Let  $n \in \mathbb{N}$  be a natural number and  $(V, d)$  a differential graded module. Then a **homotopy Poisson- $n$  algebra** over  $(V, d)$  is a family of graded linear maps*

$$D_{q_1, \dots, q_k} : \left( s^{n-2} \overline{sV^{\otimes q_1}} \right) \wedge \dots \wedge \left( s^{n-2} \overline{sV^{\otimes q_k}} \right) \rightarrow sV, \quad (48)$$

*which are homogeneous of degree  $k - n$  and indexed by the parameters  $k, q_1, \dots, q_k \in \mathbb{N}$ , with  $q_1 \leq \dots \leq q_k$  and  $D_1 = d$ . In addition this family of structure maps has to satisfy the structure equations*

$$0 = \sum_{j=1}^k \sum_{\sigma \in Sh(j, k-j)} \sum_{q_1=1}^{p_{\sigma(1)}} \dots \sum_{q_j=1}^{p_{\sigma(j)}} (-1)^{\sigma+j(k-j)} e(\sigma; v_1, \dots, v_k) \\ D_{p, p_{\sigma(j+1)}, \dots, p_{\sigma(k)}} (D_{q_1, \dots, q_j}^{p_{\sigma(1)}, \dots, p_{\sigma(j)}} (v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(j)}) \wedge v_{\sigma(j+1)} \wedge \dots \wedge v_{\sigma(k)}) \quad (49)$$

*with  $p = 1 + \sum_i (p_{\sigma(i)} - q_i)$ , for all  $k, p_1, \dots, p_k \in \mathbb{N}$  and homogeneous  $v_i \in \overline{sV^{\otimes p_i}}$ , with  $1 \leq i \leq k$ .*

## APPENDIX B. A PRIMER ON LIE RINEHART PAIRS

Lie Rinehart pairs generalize the algebraic structure of vector fields and smooth functions to commutative- and Lie-algebras, which are some kind of module with respect to each other.

According to Huebschmann [9], this idea was first used implicitly in the work of Jacobsen [10] to study certain field extensions. Further research then appeared in Herz [7], Palais [14], Rinehart [16], Huebschmann [8], Moerdijk & Mrcun [13] and now a comprehensive survey is given in [9].

They provide a purely algebraic model for the half geometric, half algebraic *Lie algebroids*, but appear more general, since there is no restriction to any notion of 'smoothness' whatsoever.

As a nontrivial 'horizontal' generalization of Lie *algebras* they have their natural interpretation in the Poisson operad (not the Lie operad), which might be the basic difference to ordinary Lie theory, when it comes to differentiation/integration.

**B.1. Lie Rinehart pairs.** After a short introduction to general (commutative) Lie Rinehart pairs we look at the special case, where the Lie algebra is a torsionless module with respect to its commutative partner. Those pairs allow for general Cartan calculus as known from multivector fields and differential forms.

In what follows, the symbol  $\mathfrak{g}$  will always mean a real Lie algebra, i.e. a  $\mathbb{R}$ -vector space together with an antisymmetric, bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \quad (50)$$

called the **Lie bracket**, such that for any three vector  $x_1, x_2$  and  $x_3 \in \mathfrak{g}$  the **Jacobi identity**  $[x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0$  is satisfied. If not stated otherwise,  $\mathfrak{g}$  will be considered as a (trivially)  $\mathbb{Z}$ -graded Lie algebra, concentrated in degree zero.

In addition  $A$  will always mean a real associative and commutative algebra with unit, that is a  $\mathbb{R}$ -vector space together with an associative and commutative, bilinear map

$$\cdot : A \times A \rightarrow A \quad (51)$$

called the product of  $A$  and a unit  $1_A \in A$ . If not stated otherwise,  $A$  will be considered as a (trivially)  $\mathbb{Z}$ -graded algebra, concentrated in degree zero.

According to a better readable text, we usually suppress the symbol of the multiplication in  $A$  and just write  $ab$  instead of  $a \cdot b$ .

Moreover  $Der(A)$  will be the Lie algebra of derivations of  $A$ , i.e. the vector space of linear endomorphisms of  $A$ , with  $D(ab) = D(a)b + aD(b)$  and Lie bracket  $[D, D'](a) := D(D'(a)) - D'(D(a))$  for any  $a, b \in A$  and  $D, D' \in Der(A)$ .

Before we get to Lie Rinehart pairs, it is handy to define Lie algebra modules first:

**Definition B.1** (Lie algebra module). *The algebra  $A$  is called a **Lie algebra module** for the Lie algebra  $\mathfrak{g}$ , if there is a morphism of Lie algebras  $D : \mathfrak{g} \rightarrow Der(A)$ . In that case,  $D$  is called the  **$\mathfrak{g}$ -scalar multiplication** on  $A$ .*

Now a *Lie Rinehart pair* is nothing but a Lie algebra together with a unital and commutative algebra, each of them being a module with respect to the other, such that a particular compatibility equation is satisfied:

**Definition B.2** (Lie Rinehart pair). *Let  $A$  be an associative and commutative algebra with unit,  $\mathfrak{g}$  a Lie algebra and  $\cdot_A : A \times \mathfrak{g} \rightarrow \mathfrak{g}$  as well as  $D : \mathfrak{g} \rightarrow \text{Der}(A)$ ;  $x \mapsto D_x$  maps, such that  $A$  is a  $\mathfrak{g}$ -module with  $\mathfrak{g}$ -scalar multiplication  $D$ , the vector space  $\mathfrak{g}$  is an  $A$ -module with  $A$ -scalar multiplication  $\cdot_A$  and the **Leibniz rule***

$$[x, a \cdot_A y] = D_x(a) \cdot_A y + a \cdot_A [x, y] \quad (52)$$

*is satisfied for any  $x, y \in \mathfrak{g}$  and  $a \in A$ . Then  $(A, \mathfrak{g})$  is called a **Lie Rinehart pair** and the map  $D$  is called its **anchor map**.*

This was first referred to as an  $(\mathbb{R}, A)$ -Lie algebra ([8] and [16]) and later J. Huebschmann called it a Lie Rinehart algebra. We use the term Lie Rinehart pair, to stress that both partners should be seen on an equal footing.

*Remark.* Judged by the fact that any Lie Rinehart pair is predominantly an algebra for the *Poisson* operad, not the Lie operad [11], this structure should eventually be called a **Poisson-Rinehart pair**, therefore pointing to the fundamental role of the Poisson structure instead of (only) the Lie structure.

The two most extreme examples derive from commutative algebras on one side and Lie algebras on the other. This reflects the fact that a Poisson algebras is a certain combination of a Lie and a commutative algebra:

*Example 8.* For any commutative and associative algebra with unit  $A$ , a Lie Rinehart pair is given by  $(A, \text{Der}(A))$ , together with the standard  $A$ -module structure of  $\text{Der}(A)$  and the identity as anchor map.

*Example 9.* Any real Lie algebra  $\mathfrak{g}$  is a  $\mathbb{R}$ -module with respect to its ordinary scalar multiplication and therefore  $(\mathbb{R}, \mathfrak{g})$  is a Lie Rinehart pair, with trivial anchor

$$D : \mathfrak{g} \times \mathbb{R} \rightarrow \mathbb{R}; (x, \lambda) \mapsto D_x(\lambda) := 0.$$

One of the most prominent examples is a particular instance of example (8). It is at the heart of calculus in differential geometry and provides the mathematical background for symplectic and multisymplectic geometry:

*Example 10.* Let  $M$  be a differentiable manifold,  $C^\infty(M)$  the algebra of smooth, real valued functions and  $\mathfrak{X}(M)$  the Lie algebra of vector fields on  $M$ .  $\mathfrak{X}(M)$  is a  $C^\infty(M)$ -module and vector fields act as derivations on smooth functions, that is the map

$$D : \mathfrak{X}(M) \times C^\infty(M) \rightarrow C^\infty(M); (X, f) \mapsto D_X(f) := X(f)$$

satisfies the equation  $D_X(fg) = D_X(f)g + fD_X(g)$ . Moreover the Leibniz rule  $[X, fY] = D_X(f)Y + f[X, Y]$  holds.

The following important example unmaskes Lie algebroids as special Lie Rinehart pairs, but expressed in a more *geometric* flavor. This is analog to the situation of projective modules and smooth vector bundles, as exhibited by the Serre-Swan theorem:

*Example 11.* A **Lie algebroid**  $(E, M, [\cdot, \cdot], D)$  is a smooth vector bundle  $E \rightarrow M$  with a Lie bracket  $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$  on its sections  $\Gamma(E)$  and a morphisms of vector bundles  $D : E \rightarrow TM$ , called the anchor, such that the tangent map  $TD$  induce a morphism of Lie algebras, with  $[X, f \cdot Y] = f \cdot [X, Y] + TD_X(f) \cdot Y$  for all  $X, Y \in \Gamma(E)$  and  $f \in C^\infty(X)$ .



With the Lie Rinehart structure at hand, their morphisms are defined as appropriate algebra maps, that interact properly with respect to the additional module structures [16]:

**Definition B.3** (Lie Rinehart Morphism). *Let  $(A, \mathfrak{g})$  and  $(B, \mathfrak{h})$  be two Lie Rinehart pairs. A **morphism of Lie Rinehart pairs** is a pair of maps  $(f, g)$ , such that  $f : A \rightarrow B$  is a morphism of associative and commutative, real algebras with unit,  $g : \mathfrak{g} \rightarrow \mathfrak{h}$  is a morphism of Lie algebras and the equations*

$$g(a \cdot_A x) = f(a) \cdot_B g(x) \quad \text{and} \quad f(D_x(a)) = D_{g(x)}(f(a)) \quad (53)$$

*are satisfied for any  $a \in A$  and  $x \in \mathfrak{g}$ .*

This is the covariant definition. If we see a Lie Rinehart pair as a Lie algebroid, the structure maps are usually defined contravariant as functions between the de Rham complexes of the algebroids.

The usability of differential geometry is build to a large extent on its computational simplicity. Often basic analysis can solve complex geometric problems. The underlying rules are *Cartan calculus*, but to make something like this available in the general Lie Rinehart setting, a certain non-degeneracy condition on the double dual of the Lie partner is necessary:

We write  $\mathfrak{g}_A^\vee := \text{Hom}_{A\text{-mod}}(\mathfrak{g}, A)$  for the  $A$ -dual of the  $A$ -module  $\mathfrak{g}$  and  $\mathfrak{g}^{\vee\vee}$  for the appropriate double dual. The following definition specializes torsionless (semi-reflexive) modules to the Lie Rinehart setting:

**Definition B.4** (Torsionless Lie Rinehart pair). *Let  $(A, \mathfrak{g})$  be a Lie Rinehart pair, such that the natural map  $\mathfrak{g} \rightarrow \mathfrak{g}^{\vee\vee} ; x \mapsto (\mathfrak{g}^\vee \rightarrow A ; f \mapsto f(x))$  is injective. Then  $(A, \mathfrak{g})$  is called **torsionless** (or **semi-reflexive**).*

Torsionless Lie Rinehart pairs hold a non degenerate and therefore unique pairing between tensors and cotensors, providing a Cartan calculus, that is completely analog to the differential geometric setting (B.4).

**B.2. Exterior tensor algebra.** We look at the exterior tensor power of the Lie algebra, seen as a module with respect to its commutative partner. The Lie bracket extends to the *Schouten-Nijenhuis bracket*, which interacts with the exterior product in terms of a Gerstenhaber structure. This is the 'free commutative prolongation'  $\text{Com}(A \oplus s\mathfrak{g})$  of the Gerstenhaber structure on  $A \oplus s\mathfrak{g}$ .

We define  $\otimes_A^0 \mathfrak{g} := A$  and write  $\otimes_A^n \mathfrak{g}$  for the  $n$ -fold  $A$ -tensor products of the  $A$ -module  $\mathfrak{g}$ . Since  $A$  is commutative,  $\otimes_A^n \mathfrak{g}$  is an  $A$ -module.

**Definition B.5** (Exterior tensor algebra). *Let  $(A, \mathfrak{g})$  be a Lie Rinehart pair and  $n \in \mathbb{Z}$ . For  $n < 0$  define  $X_n(\mathfrak{g}, A) = \{0\}$  and for  $n \geq 0$  let  $X_n(\mathfrak{g}, A) := \otimes_A^n \mathfrak{g} / J^n$  be the quotient  $A$ -module of the  $n$ -th tensor power by the submodule  $J^n$ , which is spanned from all  $x_1 \otimes \cdots \otimes x_n$  with  $x_i = x_j$  for some  $i \neq j$ . Then the direct sum*

$$X_\bullet(\mathfrak{g}, A) := \bigoplus_{n \in \mathbb{Z}} X_n(\mathfrak{g}, A) \quad (54)$$

*together with the quotient  $\wedge : X_\bullet(\mathfrak{g}, A) \times X_\bullet(\mathfrak{g}, A) \mapsto X_\bullet(\mathfrak{g}, A) ; (x, y) \mapsto x \wedge y$  of the  $A$ -tensor multiplication, is called the **exterior tensor algebra** of  $(A, \mathfrak{g})$  and the product is called the **exterior tensor product**. The induced grading on  $X_\bullet(\mathfrak{g}, A)$  is called the **tensor grading**.*

In addition we define  $X^{-n}(\mathfrak{g}, A) := X_n(\mathfrak{g}, A)$  for any integer  $n \in \mathbb{Z}$ . The induced grading on the direct sum

$$X^\bullet(\mathfrak{g}, A) := \bigoplus_{n \in \mathbb{Z}} X^n(\mathfrak{g}, A) \quad (55)$$

is called the **cotensor grading**. If the grading is irrelevant we just write  $X(\mathfrak{g}, A)$ .

*Example 12.* If  $(C^\infty(M), \mathfrak{X}(M))$  is the Lie Rinehart pair of smooth functions and vector fields, the exterior tensor algebra is the algebra of **multivector fields**  $\wedge \mathfrak{X}(M)$ .

The exterior tensor power of a Lie Rinehart pair has the structure of a *Gerstenhaber algebra* (also called Poisson-2 algebra) with respect to the exterior product and the Schouten-Nijenhuis bracket. The latter is nothing but the free commutative lift of the Lie bracket on the Gerstenhaber algebra  $A \oplus \mathfrak{sg}$ .

**Definition B.6** (Schouten-Nijenhuis bracket). *Let  $(A, \mathfrak{g})$  be a Lie Rinehart pair with exterior tensor power  $X(\mathfrak{g}, A)$ . Then the **Schouten Nijenhuis bracket** is the map*

$$[\cdot, \cdot] : X(\mathfrak{g}, A) \times X(\mathfrak{g}, A) \rightarrow X(\mathfrak{g}, A), \quad (56)$$

defined by  $[a, b] = 0$  as well as  $[x, a] = [a, x] = D_x(a)$  on scalars  $a, b \in A$  and vectors  $x \in \mathfrak{g}$  and by

$$\begin{aligned} [x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_m] = \\ \sum_{i,j} (-1)^{i+j} [x_i, y_j] \wedge x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots \wedge \widehat{y}_j \wedge \cdots \wedge y_m \end{aligned} \quad (57)$$

on simple tensors  $x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_m \in X(\mathfrak{g}, A)$  and then extend to  $X(\mathfrak{g}, A)$  by  $A$ -additivity.

We write  $(X(\mathfrak{g}, A), \wedge, [\cdot, \cdot])$  for the appropriate Gerstenhaber algebra and call it the **Schouten-Nijenhuis algebra** of  $(A, \mathfrak{g})$ .

In case of multivector fields, a proof can be found for example in [12] or [?]. Those proofs are an implication of the Lie Rinehart structure only and therefore carry over verbatim to the general situation.

**B.3. The de Rham complex.** We generalize the de Rham complex of differential forms to the setting of a torsionless Lie Rinehart pair.

Recall that we consider  $\mathfrak{g}$  as an  $A$ -module but concentrated in cotensor degree  $-1$ . Therefore the dual module  $\mathfrak{g}^\vee$  is an  $A$ -module concentrated in cotensor degree  $1$ . We put  $\otimes_A^0 \mathfrak{g}_A^\vee := A$  (concentrated in degree zero) and write  $\otimes_A^n \mathfrak{g}_A^\vee$  for the  $n$ -fold graded  $A$ -tensor products of the  $A$ -module  $\mathfrak{g}^\vee$ . Since  $A$  is commutative, each  $\otimes_A^n \mathfrak{g}_A^\vee$  is an  $A$ -module.

**Definition B.7** (Exterior cotensor algebra). *Let  $(A, \mathfrak{g})$  be a Lie Rinehart pair and  $n \in \mathbb{Z}$ . For  $n < 0$  define  $\Omega^n(\mathfrak{g}, A) = \{0\}$  and if  $n \geq 0$  let  $\Omega^n(\mathfrak{g}, A) := \otimes_A^n \mathfrak{g}_A^\vee / J^n$  be the quotient  $A$ -module of the  $n$ -th tensor product of the dual module and the submodule  $J^n$ , spanned by all cotensors  $x^1 \otimes \cdots \otimes x^n$  with  $x^i = x^j$  for some  $i \neq j$ . Then the direct sum*

$$\Omega^\bullet(\mathfrak{g}, A) = \bigoplus_{n \in \mathbb{Z}} \Omega^n(\mathfrak{g}, A) \quad (58)$$

together with the quotient  $\wedge : \Omega^\bullet(\mathfrak{g}, A) \times \Omega^\bullet(\mathfrak{g}, A) \rightarrow \Omega^\bullet(\mathfrak{g}, A)$ ;  $(x, y) \mapsto x \wedge y$  of the  $A$ -cotensor multiplication, is called the **exterior cotensor algebra** of  $(A, \mathfrak{g})$  and the product is called the **exterior cotensor product**. The grading on  $\Omega^\bullet(\mathfrak{g}, A)$  is called the **cotensor grading**.

In addition we define  $\Omega_n(\mathfrak{g}, A) := \Omega^{-n}(\mathfrak{g}, A)$  for any integer  $n \in \mathbb{Z}$ . The induced grading on the direct sum

$$\Omega_\bullet(\mathfrak{g}, A) := \bigoplus_{n \in \mathbb{Z}} \Omega_n(\mathfrak{g}, A) \quad (59)$$

is called the **tensor grading**. If the grading is irrelevant we just write  $\Omega(\mathfrak{g}, A)$ .

The exterior cotensor power has the structure of a differential graded algebra, at least if the Lie Rinehart pair is torsionless. This refines the well known de Rham Complex of differential forms to the general torsionless Lie Rinehart setting:

**Definition B.8** (De Rham differential). *Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair with exterior cotensor algebra  $\Omega(\mathfrak{g}, A)$ . Then the **de Rham differential** (or exterior differential)*

$$d : \Omega(\mathfrak{g}, A) \rightarrow \Omega(\mathfrak{g}, A)$$

is defined for a homogeneous cotensor  $f \in \Omega^k(\mathfrak{g}, A)$  and any simple exterior tensors  $x_0 \wedge \cdots \wedge x_k \in X(\mathfrak{g}, A)$  by the equation

$$\begin{aligned} df(x_0 \wedge \cdots \wedge x_k) &= \sum_j (-1)^j D_{x_j}(f(x_0 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_k)) \\ &\quad + \sum_{i < j} (-1)^{i+j} f([x_i, x_j] \wedge x_0 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_k) \end{aligned} \quad (60)$$

and is then extended to all of  $\Omega(\mathfrak{g}, A)$  by  $A$ -linearity.

We call both the chain complex  $(\Omega^\bullet(\mathfrak{g}, A), d)$  as well as the cochain complex  $(\Omega_\bullet(\mathfrak{g}, A), d)$ , the **de Rham complex** of a Lie Rinehart pair.

Since the natural pairing is non degenerate, the exterior derivative is uniquely defined by equation (60). It is moreover a graded map, homogeneous of degree 1 with respect to the cotensor grading (hence of degree  $-1$  with respect to the tensor grading). Moreover  $d$  is a differential, that is  $d^2 = 0$ .

*Example 13.* If the Lie Rinehart pair happens to be a Lie algebra  $(\mathfrak{g}, A)$ , the de Rham complex is equal to the Chevalley-Eilenberg complex of that Lie algebra. It should however be noted, that in general the de Rham complex of a Lie Rinehart pair is *not* equivalent to the Chevalley-Eilenberg complex of just the Lie partner.

**B.4. Cartan Calculus.** We close this section with an introduction to the basic computational rules of Cartan calculus. These rules are simple yet powerful tools, which we use extensively in the present work.

The sceptical reader should pay special attention to the last equation in (72). To the best of the authors knowledge, this equation does not appear anywhere else, but is of central importance in this work.

To start, observe that the dual nature of tensors and cotensors gives rise to a bilinear map, that basically *evaluates* cotensors along tensors of equal degree:

**Definition B.9** (Natural Pairing). *Let  $(A, \mathfrak{g})$  be a Lie Rinehart pair. The graded  $A$ -linear map*

$$\langle \cdot, \cdot \rangle : \Omega^\bullet(\mathfrak{g}, A) \otimes_A X^\bullet(\mathfrak{g}, A) \rightarrow A, \quad (61)$$

defined by  $\langle a, b \rangle = ab$  on scalars  $a, b \in A$ , by  $\langle f, x \rangle = 0$  on cotensors  $f \in \Omega^p(\mathfrak{g}, A)$  and tensors  $x \in X^q(\mathfrak{g}, A)$ , such that  $p + q \neq 0$  and by

$$\langle f_1 \wedge \cdots \wedge f_k, x_1 \wedge \cdots \wedge x_k \rangle = \sum_{s \in S_k} (-1)^s \langle f_{s(1)}, x_1 \rangle \cdots \langle f_{s(k)}, x_k \rangle \quad (62)$$

on simple cotensors  $f_1 \wedge \cdots \wedge f_k \in \Omega^\bullet(\mathfrak{g}, A)$  and tensors  $x_1 \wedge \cdots \wedge x_k \in X^\bullet(\mathfrak{g}, A)$  and then extended to all of  $\Omega^\bullet(\mathfrak{g}, A) \otimes_A X^\bullet(\mathfrak{g}, A)$  by  $A$ -linearity, is called the **natural pairing**.

Since the determinant is invariant with respect to transposition, the defining equation can equivalently be written in its transposed version

$$\langle f_1 \wedge \cdots \wedge f_m, x_1 \wedge \cdots \wedge x_m \rangle = \sum_{s \in S_m} (-1)^s \langle f_1, x_{s(1)} \rangle \cdots \langle f_m, x_{s(m)} \rangle. \quad (63)$$

The following proposition shows that the natural pairing is non-degenerate and therefore unique, in case the Lie Rinehart pair is torsionless. Since all operators in Cartan calculus are defined in terms of this pairing, it justifies our restriction to the torsionless setting:

**Proposition B.10.** *Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair with natural pairing  $\langle \cdot, \cdot \rangle : \Omega^\bullet(\mathfrak{g}, A) \otimes_A X^\bullet(\mathfrak{g}, A) \rightarrow A$  and  $x, y \in \mathfrak{g}$  two vectors. Then  $\langle f, x \rangle = \langle f, y \rangle$  for all  $f \in \mathfrak{g}^\vee$  implies  $x = y$ .*

*Proof.* Since  $(A, \mathfrak{g})$  is torsionless, the inclusion  $\mathfrak{g} \rightarrow \mathfrak{g}^{\vee\vee}$ ;  $x \mapsto (\mathfrak{g}^\vee \rightarrow A; f \mapsto f(x))$  is injective. Hence  $f(x) = f(y)$  for all  $f \in \mathfrak{g}^\vee$  implies  $x = y$ .  $\square$

With a non degenerate pairing, contraction can be defined consistently by a certain adjointness property. In fact we determine the usual right contraction of a cotensor along a tensor as right adjoint to the exterior product and dually the left contraction of a tensor by a cotensor as left adjoint to the exterior product.

The latter operator seems to be not widely known in the literature. However it appears in [12] for multivector fields and differential forms and we generalize it to arbitrary torsionless Lie Rinehart pairs:

**Definition B.11** (Right/Left Contraction). *Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair and  $x \in X_{-k}(\mathfrak{g}, A) \simeq X^k(\mathfrak{g}, A)$  a tensor homogeneous of cotensor degree  $k$ . Then the **right contraction along**  $x$  is the map*

$$i_x : \Omega^\bullet(\mathfrak{g}, A) \rightarrow \Omega^\bullet(\mathfrak{g}, A),$$

*that is defined for any integer  $n \in \mathbb{Z}$  and homogeneous cotensor  $f \in \Omega^n(\mathfrak{g}, A)$  by the equation*

$$\langle i_x f, y \rangle = \langle f, x \wedge y \rangle \quad (64)$$

*for all  $y \in X^\bullet(\mathfrak{g}, A)$  and is then extended to  $\Omega^\bullet(\mathfrak{g}, A)$  by additivity.*

*Dually for any cotensor  $f \in \Omega^k(\mathfrak{g}, A) \simeq \Omega_{-k}(\mathfrak{g}, A)$ , homogeneous of cotensor degree  $k$ , the **left contraction along**  $f$  is the map*

$$j_f : X^\bullet(\mathfrak{g}, A) \rightarrow X^\bullet(\mathfrak{g}, A)$$

*that is defined for any integer  $n \in \mathbb{Z}$  and homogeneous tensor  $x \in X^n(\mathfrak{g}, A)$  by the equation*

$$\langle g, j_f x \rangle = \langle g \wedge f, x \rangle \quad (65)$$

*for all  $g \in \Omega^\bullet(\mathfrak{g}, A)$  and then extended to  $\Omega^\bullet(\mathfrak{g}, A)$  by additivity.*

The following corollary summarizes the basic rules, regarding the left and right contraction operators:

**Corollary B.12.** *Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair,  $x, y \in X^\bullet(\mathfrak{g}, A)$  and  $f, g \in \Omega^\bullet(\mathfrak{g}, A)$ . Then*

$$i_{x \wedge y} f = i_y \circ i_x f \quad \text{and} \quad j_{f \wedge g} x = j_f \circ j_g x. \quad (66)$$

*If  $x$  and  $f$  are homogeneous, the contractions  $i_x f$  and  $j_f x$  are homogeneous, with  $|i_x f| = |x| + |f|$  and  $|j_f x| = |x| + |f|$ .*

*Proof.* The first two equations follow from the associativity of the exterior products. In particular, let  $z \in X^\bullet(A, \mathfrak{g})$  be any tensor. Then

$$\begin{aligned} \langle i_y \circ i_x f, z \rangle &= \langle i_x f, y \wedge z \rangle \\ &= \langle f, x \wedge y \wedge z \rangle \\ &= \langle i_{x \wedge y} f, z \rangle. \end{aligned}$$

Hence the first equation follows from (B.10). The computation of the second equation is analog.

To see the homogeneity part, observe that we consider tensors as concentrated in negative degrees and cotensors as concentrated in positive degrees, with respect to the cotensor grading.  $\square$

Since the contraction operators are linear in both arguments, we can introduce their kernels the usual way:

**Definition B.13** (Kernel). *Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair. The **kernel** of a cotensor  $f \in \Omega^\bullet(\mathfrak{g}, A)$  is the set*

$$\ker(f) := \{x \in X^\bullet(\mathfrak{g}, A) \mid i_x f = 0\}$$

*of all tensors, such that the right contraction of  $f$  along that tensor vanishes and the **kernel** of a tensor  $x \in X^\bullet(\mathfrak{g}, A)$  is the set*

$$\ker(x) := \{f \in \Omega^\bullet(\mathfrak{g}, A) \mid j_f x = 0\}$$

*of all cotensors, such that the left contraction of  $x$  along that cotensor vanishes.*

The Lie derivative along tensors is now defined as usual in terms of Cartan's infinitesimal homotopy formula:

**Definition B.14.** *Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair and  $x \in X^\bullet(\mathfrak{g}, A)$  a homogeneous tensor. Then the map*

$$L_x : \Omega^\bullet(\mathfrak{g}, A) \rightarrow \Omega^\bullet(\mathfrak{g}, A) ; f \mapsto di_x f - (-1)^{|x|} i_x df \quad (67)$$

*is called the **Lie derivative** along  $x$  and equation (67) is called **Cartan's infinitesimal graded homotopy formula**.*

The following proposition shows the basic computation rules that we need in what follows. Most of these equations are long known. However to the best of the authors knowledge it seems that the last equation does not appear anywhere else in the literature.

**Proposition B.15.** *Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair,  $x, y \in X_\bullet(\mathfrak{g}, A)$  homogeneous tensors and  $f \in \Omega_\bullet(\mathfrak{g}, A)$  a cotensor. Then*

$$dL_x f = (-1)^{|x|-1} L_x df \quad (68)$$

$$i_{[x,y]} f = (-1)^{(|x|-1)|y|} L_x i_y f - i_y L_x f \quad (69)$$

$$L_{[x,y]} f = (-1)^{(|x|-1)(|y|-1)} L_x L_y f - L_y L_x f \quad (70)$$

$$L_{x \wedge y} f = (-1)^{|y|} i_y L_x f + L_y i_x f \quad (71)$$

$$i_{j(i_y f) x} f = i_y f \wedge i_x f \quad (72)$$

*Proof.* Except for (72), these equations are long known. A proof is given in [3] in the context of multivector fields and differential forms, but can be carried over verbatim into the general Lie Rinehart setting. My proof of (72) is verly long and therefore done separately in the next section.  $\square$

**B.4.1. Proof of Equation (72).** This section is devoted to the proof of equation (72). Unfortunately I was not able to derive it easily, but only by a direct and very long computation.

To start, we derive explicit expressions for the left and right contraction operations, in case both arguments are simple:

**Proposition B.16.** *Given a torsionless Lie Rinehart pair  $(A, \mathfrak{g})$ , a simple tensor  $x := x_1 \wedge \cdots \wedge x_r \in X(\mathfrak{g}, A)$  and a simple cotensor  $f := f_1 \wedge \cdots \wedge f_m \in \Omega(\mathfrak{g}, A)$ , the right contraction  $i_x f$  vanishes for  $r > m$  and can be computed by*

$$i_x f = \frac{1}{(m-r)!} \sum_{\sigma \in S_m} (-1)^\sigma \langle f_{\sigma(1)}, x_1 \rangle \cdots \langle f_{\sigma(r)}, x_r \rangle \cdot f_{\sigma(r+1)} \wedge \cdots \wedge f_{\sigma(m)} \quad (73)$$

*if  $r \leq m$ . Similar the left contraction  $j_f x$  vanishes for  $m > r$  and can be computed by*

$$j_f x = \frac{1}{(r-m)!} \sum_{\sigma \in S_r} (-1)^\sigma \langle f_1, x_{\sigma(r-m+1)} \rangle \cdots \langle f_m, x_{\sigma(r)} \rangle \cdot x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(r-m)}. \quad (74)$$

*if  $m \leq r$ .*

*Proof.* Both equations are direct consequences of the defining identities for the contraction operators (65), (64) and the natural pairing (62). To be more precise we use the following arguments:

To see that the right contraction vanishes for  $r > m$  observe that the defining equation  $\langle i_x f, y \rangle = \langle f, x \wedge y \rangle$  for all  $y \in X(\mathfrak{g}, A)$  implies  $i_x f = 0$ , since either  $|x \wedge y| > |f|$ , or  $x \wedge y = 0$  and the natural pairing is zero in both cases.

To show equation (73) for any  $r \leq m$ , let  $y_1 \wedge \cdots \wedge y_{m-r} \in X(\mathfrak{g}, A)$  be an arbitrary simple tensor of degree  $(m-r)$ . Then substitute (73) into the left side of (64) to get

$$\begin{aligned} \langle i_{x_1} \wedge \cdots \wedge x_r f_1 \wedge \cdots \wedge f_m, y_1 \wedge \cdots \wedge y_{m-r} \rangle = \\ \frac{1}{(m-r)!} \sum_{\sigma \in S_m} (-1)^\sigma \langle f_{\sigma(1)}, x_1 \rangle \cdots \langle f_{\sigma(r)}, x_r \rangle \cdot \\ \cdot \langle f_{\sigma(r+1)} \wedge \cdots \wedge f_{\sigma(m)}, y_1 \wedge \cdots \wedge y_{m-r} \rangle. \end{aligned}$$

Then apply (62) to the most right pairing on the right side. This can be written as follows: (According to a better readable text we write  $S_M$  for the set of all permutations defined explicit on the finite set  $M$ )

$$\begin{aligned} \frac{1}{(m-r)!} \sum_{\sigma \in S_m} \sum_{\tau \in S_{\{\sigma(r+1), \dots, \sigma(m)\}}} (-1)^\sigma (-1)^\tau \cdot \\ \cdot \langle f_{\sigma(1)}, x_1 \rangle \cdots \langle f_{\sigma(r)}, x_r \rangle \cdot \langle f_{\sigma(r+1)}, y_1 \rangle \cdots \langle f_{\sigma(m)}, y_{m-r} \rangle. \quad (75) \end{aligned}$$

Now observe, that for any  $\sigma \in S_m$  and  $\tau \in S_{\{\sigma(r+1), \dots, \sigma(m)\}}$ , the permutation  $(\sigma(1), \dots, \sigma(r), \tau\sigma(r+1), \dots, \tau\sigma(m))$  is again an element of  $S_m$  and since there are precisely  $(m-r)!$  many permutations in  $S_{\{\sigma(r+1), \dots, \sigma(m)\}}$ , we can just 'absorb' the sum over  $S_{\{\sigma(r+1), \dots, \sigma(m)\}}$  in (75) into a sum over  $S_m$  and therefore (75) is the same as

$$\frac{1}{(m-r)!} \sum_{\sigma \in S_m} (-1)^\sigma \langle f_{\sigma(1)}, x_1 \rangle \cdots \langle f_{\sigma(r)}, x_r \rangle \cdot \langle f_{\sigma(r+1)}, y_1 \rangle \cdots \langle f_{\sigma(m)}, y_{m-r} \rangle. \quad (76)$$

Using the definition of the natural pairing (62), this transforms expression (76) into

$$\langle f_1 \wedge \cdots \wedge f_m, x_1 \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_{m-r} \rangle$$

and therefore proofs (64) for arbitrary simple tensors  $y_1 \wedge \cdots \wedge y_{m-r}$  and hence for any tensor of degree  $(m-r)$ , since (64) is  $A$ -linear in both arguments and any such tensor is a sum of simple ones. For tensor  $y \in X(\mathfrak{g}, A)$  of degree other than  $(m-r)$  both sides of equation (64) vanish and this proves equation (73).

The proof for the second equation is analog, taking the 'left nature' of the defining equation (65) into account.

To see that the left contraction vanishes for  $m > r$ , observe that  $\langle g, j_fx \rangle = \langle g \wedge f, x \rangle$  for all  $g \in \Omega(\mathfrak{g}, A)$  implies  $j_fx = 0$ , since either  $|g \wedge f| > |x|$ , or  $g \wedge f = 0$  and the natural pairing is zero in both cases.

To see that the left contraction operator satisfies equation (74) for any  $r \leq m$ , let  $g_1 \wedge \cdots \wedge g_{r-m} \in \Omega(\mathfrak{g}, A)$  be any arbitrary simple cotensor of degree  $(r-m)$ . Substitute (74) into the left side of (65) to get

$$\begin{aligned} \langle g_1 \wedge \cdots \wedge g_{r-m}, j_{f_1 \wedge \cdots \wedge f_m} x_1 \wedge \cdots \wedge x_r \rangle = \\ \frac{1}{(r-m)!} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma \langle f_1, x_{\sigma(r-m+1)} \rangle \cdots \langle f_m, x_{\sigma(r)} \rangle \cdot \\ \cdot \langle g_1 \wedge \cdots \wedge g_{r-m}, x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(r-m)} \rangle \end{aligned}$$

and apply the transposed identity (63) to the most right pairing on the right side. This can be written as follows: (According to a better readable text we write  $\mathbb{S}_M$  for the set of all permutations defined explicit on the finite set  $M$ .)

$$\begin{aligned} \frac{1}{(r-m)!} \sum_{\sigma \in \mathbb{S}_r} \sum_{t \in \mathbb{S}_{\{\sigma(1), \dots, \sigma(r-m)\}}} (-1)^\sigma (-1)^\tau \cdot \\ \cdot \langle f_1, x_{\sigma(r-m+1)} \rangle \cdots \langle f_m, x_{\sigma(r)} \rangle \cdot \langle g_1, x_{\tau(\sigma(1))} \rangle \cdots \langle g_{r-m}, x_{\tau(\sigma(r-m))} \rangle \end{aligned}$$

Now observe, that for any  $\sigma \in \mathbb{S}_r$  and  $\tau \in \mathbb{S}_{\{\sigma(1), \dots, \sigma(r-m)\}}$ , the permutation  $(\tau(\sigma(1)), \dots, \tau(\sigma(r-m)), \sigma(r-m+1), \dots, \sigma(r))$  is again an element of  $\mathbb{S}_r$  and since there are precisely  $(r-m)!$  many permutations in  $\mathbb{S}_{\{\sigma(1), \dots, \sigma(r-m)\}}$ , we can just 'absorb' the sum over  $\mathbb{S}_{\{\sigma(1), \dots, \sigma(r-m)\}}$  in the previous expression into the sum over  $\mathbb{S}_r$ . This gives

$$\begin{aligned} \sum_{\sigma \in \mathbb{S}_r} (-1)^\sigma \langle f_1, x_{\sigma(r-m+1)} \rangle \cdots \langle f_m, x_{\sigma(r)} \rangle \langle g_1, x_{\sigma(1)} \rangle \cdots \langle g_{r-m}, x_{\sigma(r-m)} \rangle = \\ \langle f_1 \wedge \cdots \wedge f_m \wedge g_1 \wedge \cdots \wedge g_{r-m}, x_{r-m+1} \wedge \cdots \wedge x_r \wedge x_1 \wedge \cdots \wedge x_{r-m} \rangle = \\ (-1)^{m(r-m)+m(r-m)} \langle g_1 \wedge \cdots \wedge g_{r-m} \wedge f_1 \wedge \cdots \wedge f_m, x_1 \wedge \cdots \wedge x_r \rangle \end{aligned}$$

This proves (65) for arbitrary simple cotensors  $g_1 \wedge \cdots \wedge g_{r-m}$  and hence for any cotensor of degree  $(r-m)$ , since (65) is  $A$ -linear in both arguments and any such cotensor is a sum of simple ones. For cotensor  $g \in \Omega(\mathfrak{g}, A)$  of degree other than  $(r-m)$  both sides of equation (65) vanish.  $\square$

With these precomputations we can now go on and prove equation (72) by a long but very basic combinatorial exercise.

**Proposition B.17.** *Let  $(A, \mathfrak{g})$  be a torsionless Lie Rinehart pair,  $x, y \in X(\mathfrak{g}, A)$  homogeneous tensors and  $f \in \Omega(\mathfrak{g}, A)$  a homogeneous cotensor. Then*

$$i_{j(i_y f)} x f = i_y f \wedge i_x f. \quad (77)$$

*Proof.* To see that the equation is well defined, it is enough to show that both sides are homogeneous of the same tensor degree. Therefore recall that we consider tensors as concentrated in positive degrees and cotensors as concentrated in negative degrees. Then

$$\begin{aligned}
|i_{j(i_y f)} x f| &= |f| + |j i_y f x| \\
&= |f| + (|x| + |i_y f|) \\
&= (|f| + |y|) + (|f| + |x|) \\
&= |i_y f| + |i_x f| \\
&= |i_y f \wedge i_x f|.
\end{aligned}$$

To proof the equation it is enough to consider simple (co)tensors only. The general situation then follows since all involved operations are  $A$ -linear in all arguments and since any tensor is a (not necessarily unique) sum of simple tensors.

We write  $\langle x, f \rangle$  simultaneously for the natural pairing, the left contraction  $i_x f$ , or the right contraction  $j_f x$ , if  $|x| \leq 1$  and if  $-|f| \leq 1$ . In particular this means  $\langle x, f \rangle = x \cdot f$ , if at least one argument is an element of  $A$  (has degree zero). This reduces the overall complexity of the proof, since we don't have to distinguish cases where at least one argument is an element of  $A$ .

To proceed let  $r, k, m \in \mathbb{N}$  be natural numbers and for all  $1 \leq i \leq r$ ,  $1 \leq j \leq k$  and  $1 \leq h \leq m$  let  $x_i, y_j \in A \cup \mathfrak{g}$  be any scalar or vector and let  $f_h \in A \cup \mathfrak{g}^\vee$  be any scalar or covector. Therefore  $x_1 \wedge \cdots \wedge x_r$  and  $y_1 \wedge \cdots \wedge y_k$  are simple tensors and  $f_1 \wedge \cdots \wedge f_m$  is a simple cotensor.

We first look at the right side of equation (77). Due to (73) it vanishes whenever  $k > m$  or  $r > m$  and for  $k \leq m$  and  $r \leq m$  we can rewrite it into the more explicit form

$$\begin{aligned}
&(i_{y_1 \wedge \cdots \wedge y_k} f_1 \wedge \cdots \wedge f_m) \wedge (i_{x_1 \wedge \cdots \wedge x_r} f_1 \wedge \cdots \wedge f_m) = \\
&\quad \left( \frac{1}{(m-k)!} \sum_{\sigma \in S_m} (-1)^\sigma \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \right) \wedge \\
&\quad \left( \frac{1}{(m-r)!} \sum_{\tau \in S_m} (-1)^\tau \langle f_{\tau(1)}, x_1 \rangle \cdots \langle f_{\tau(r)}, x_r \rangle f_{\tau(r+1)} \wedge \cdots \wedge f_{\tau(m)} \right). \quad (78)
\end{aligned}$$

Due to the antisymmetry of the exterior product, this vanishes if  $r+k < m$ . To see that observe  $\{\sigma(k+1), \dots, \sigma(m)\} \cap \{\tau(r+1), \dots, \tau(m)\} \neq \emptyset$ , for  $r+k < m$  and any permutations  $\sigma, \tau \in S_m$ . Hence  $(f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)}) \wedge (f_{\tau(r+1)} \wedge \cdots \wedge f_{\tau(m)}) = 0$ .

In conclusion we can say that the right side of the equation is only nontrivial in a narrow range of tensor degrees, namely for  $r+k \geq m$  and  $r \leq m$  as well as  $k \leq m$ .

Now suppose  $r+k \geq m$ ,  $r \leq m$  and  $k \leq m$ . Since the product in  $A$  is commutative we rewrite (78) further:

$$\begin{aligned}
&\frac{1}{(m-k)!(m-r)!r!} \sum_{\sigma, v \in S_m} \sum_{\tau \in S_r} (-1)^{\sigma+v} \cdot \\
&\quad \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \langle f_{v(\tau(1))}, x_{\tau(1)} \rangle \cdots \langle f_{v(\tau(r))}, x_{\tau(r)} \rangle \cdot \\
&\quad \cdot f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)} = \\
&\frac{1}{(m-k)!(m-r)!r!} \sum_{\sigma, v \in S_m} \sum_{\tau \in S_r} (-1)^{\sigma+v+\tau} \cdot \\
&\quad \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r)}, x_{\tau(r)} \rangle \cdot
\end{aligned}$$



$$\cdot f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)} . \quad (79)$$

Due to the antisymmetry of the exterior product only terms for those permutations  $\sigma, v \in S_m$  with  $\{\sigma(k+1), \dots, \sigma(m)\} \cap \{v(r+1), \dots, v(m)\} = \emptyset$  are non zero here and since we assume  $(m-k) \leq r$  we can restrict the summation in (79) to those  $\sigma, v \in S_m$  satisfying  $\{\sigma(k+1), \dots, \sigma(m)\} \subset \{v(1), \dots, v(r)\}$ . Writing

$$I_1(m, k, r) = \{(\sigma, v) \in S_m \times S_m \mid \{\sigma(k+1), \dots, \sigma(m)\} \subset \{v(1), \dots, v(r)\}\}$$

for the appropriate index set of restricted permutations, (79) rewrites into

$$\begin{aligned} & \frac{1}{(m-k)!(m-r)!r!} \sum_{\sigma, v \in I_1(m, k, r)} \sum_{\tau \in S_r} (-1)^{\sigma+v+\tau} \cdot \\ & \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r)}, x_{\tau(r)} \rangle \cdot \\ & \cdot f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)} . \end{aligned} \quad (80)$$

To simplify this further, observe that for any  $(\sigma, v) \in I_1(m, k, r)$  there is a unique permutation,  $\phi \in S_r$ , which 'unshuffles'  $\sigma(k+1), \dots, \sigma(m)$  in  $\{v(1), \dots, v(r)\}$  to the left', i.e. which acts by

$$\begin{aligned} v \circ (\phi \times id_{m-r})(\{1, \dots, m\}) = \\ \{\sigma(k+1), \dots, \sigma(m), v\phi(m-k+1), \dots, v\phi(r), v(r+1), \dots, v(m)\} , \end{aligned}$$

where  $id_{m-r} \in S_{m-r}$  is the identity permutation, and where the order in the (ordered) set  $\{v(1), \dots, v(r)\} \setminus \{\sigma(k+1), \dots, \sigma(m)\}$  is not changed by  $\phi$ . Using this we can use the commutativity of the multiplication in  $A$ , to reorder the scalar factors. In particular we get

$$\begin{aligned} & \sum_{\tau \in S_r} (-1)^{\sigma+v+\tau} \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r)}, x_{\tau(r)} \rangle \cdot \\ & \cdot f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)} = \\ & \sum_{\tau \in S_r} (-1)^{\sigma+v(\phi \times id_{m-r})+\tau\phi} \\ & \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \langle f_{v\phi(1)}, x_{\tau\phi(1)} \rangle \cdots \langle f_{v\phi(r)}, x_{\tau\phi(r)} \rangle \cdot \\ & \cdot f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)} = \\ & \sum_{\tau \in S_r} (-1)^{\sigma+v+\tau} \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \langle f_{\sigma(k+1)}, x_{\tau(1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(m-k)} \rangle \cdot \\ & \langle f_{v\phi(m-k+1)}, x_{\tau(m-k+1)} \rangle \cdots \langle f_{v\phi(r)}, x_{\tau(r)} \rangle \cdot \\ & \cdot f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)} \end{aligned}$$

The number of ways  $\{\sigma(k+1), \dots, \sigma(m)\}$  can be a subset of  $\{v(1), \dots, v(r)\}$  is the same as counting the number of ways to put  $(m-k)$  labeled balls into  $r$  labeled slots with exclusion, which is computed by the *falling factorial*

$$P(r, m-k) = r \cdot (r-1) \cdots (r-(m-k)+1) = \frac{r!}{(r+k-m)!}$$

and if we define another index set

$$J_1(m, k, r) = \{(\sigma, v) \in S_m \times S_m \mid v(1) = \sigma(k+1), \dots, v(m-k) = \sigma(m)\}$$

we finally can rewrite (80) into

$$\begin{aligned} & \frac{1}{(m-k)!(m-r)!(r+k-m)!} \sum_{\sigma, v \in J_1(m, r, k)} \sum_{\tau \in S_r} (-1)^{\sigma+v+\tau} \cdot \\ & \quad \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \langle f_{\sigma(k+1)}, x_{\tau(1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(m-k)} \rangle \cdot \\ & \quad \langle f_{v(m-k+1)}, x_{\tau(m-k+1)} \rangle \cdots \langle f_{v(r)}, x_{\tau(r)} \rangle \cdot \\ & \quad \cdot f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)}. \end{aligned} \quad (81)$$

In the next step we transform the left side of equation (77), exactly into (81). We start by making the nested contraction  $i_{j_{i_y} f} x f$  explicit using (73) and (74). For  $k > m$ , the contraction  $i_y f$  vanishes and we assume  $k \leq m$  to compute

$$\begin{aligned} & i_{y_1 \wedge \cdots \wedge y_k} f_1 \wedge \cdots \wedge f_m = \\ & \quad \frac{1}{(m-k)!} \sum_{\sigma \in S_m} (-1)^\sigma \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)}. \end{aligned} \quad (82)$$

For  $r+k < m$ , the double contraction  $j_{i_y} f x$  is zero and therefore we assume  $r+k \geq m$  to compute

$$\begin{aligned} & j_{(i_{y_1 \wedge \cdots \wedge y_k} f_1 \wedge \cdots \wedge f_m)} x_1 \wedge \cdots \wedge x_r = \\ & \quad \frac{1}{(m-k)!} \sum_{\sigma \in S_m} (-1)^\sigma \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle j_{f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)}} x_1 \wedge \cdots \wedge x_r = \\ & \quad \frac{1}{(m-k)!(r+k-m)!} \sum_{\sigma \in S_m} \sum_{\tau \in S_r} (-1)^{\sigma+\tau} \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \\ & \quad \cdot \langle f_{\sigma(k+1)}, x_{\tau(r+k-m+1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(r)} \rangle x_{\tau(1)} \wedge \cdots \wedge x_{\tau(r+k-m)}. \end{aligned}$$

For  $r+k > 2m$ , the contraction  $j_{(i_{y_1 \wedge \cdots \wedge y_k} f_1 \wedge \cdots \wedge f_m)} x f$  is zero and therefore we assume  $r+k \leq 2m$  to compute

$$\begin{aligned} & \frac{1}{(m-k)!(r+k-m)!(2m-r-k)!} \sum_{\sigma, v \in S_m} \sum_{\tau \in S_r} (-1)^{\sigma+\tau+v} \cdot \\ & \quad \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \langle f_{\sigma(k+1)}, x_{\tau(r+k-m+1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(r)} \rangle \cdot \\ & \quad \cdot \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r+k-m)}, x_{\tau(r+k-m)} \rangle f_{v(r+k-m+1)} \wedge \cdots \wedge f_{v(m)}. \end{aligned} \quad (83)$$

To simplify this, we show that for arbitrary but fixed permutations  $\sigma, v \in S_m$  with  $\{\sigma(k+1), \dots, \sigma(m)\} \cap \{v(1), \dots, v(r+k-m)\} \neq \emptyset$  the summation over all permutations from  $S_r$  vanishes in (83).

To see that, let  $\sigma, v \in S_m$  with  $\{\sigma(k+1), \dots, \sigma(m)\} \cap \{v(1), \dots, v(r+k-m)\} \neq \emptyset$ . Then there is a  $k+1 \leq j \leq m$  and a  $1 \leq i \leq r-m+k$ , such that  $\sigma(j) = v(i)$  and for any permutation  $\tau \in S_r$  precisely one other permutation  $\tau' \in S_r$ , which is equal to  $\tau$  but with  $\tau(i)$  and  $\tau(r-m+j)$  transposed. Moreover, since  $i \neq r-m+j$  this transposition is not the identity.

Then  $\langle f_{\sigma(j)}, x_{\tau(r-m+j)} \rangle = \langle f_{v(i)}, x_{\tau'(i)} \rangle$  and  $\langle f_{\sigma(j)}, x_{\tau'(r-m+j)} \rangle = \langle f_{v(i)}, x_{\tau(i)} \rangle$  and since  $(-1)^{\tau'} = -(-1)^\tau$ , the term for  $\tau$  cancel against the term for  $\tau'$ .

Consequently we can restrict the summation in expression (83) to those  $\sigma, v \in S_m$  satisfying  $\{\sigma(k+1), \dots, \sigma(m)\} \subset \{v(r+k-m+1), \dots, v(m)\}$ .

From this follows that the left side is non zero only if  $m-k \leq 2m-r-k$ , which means  $r \leq m$ . Summarizing this, we can say that the left side of equation (83) is only nontrivial in a narrow range of tensor degrees, namely for  $r+k \geq m$  and  $r \leq m$  as well as  $k \leq m$ , which coincides with the triviality of the right side.

To proceed we assume,  $r+k \geq m$  and  $r \leq m$  as well as  $k \leq m$  and write

$$I_2(m, k, r) = \{(\sigma, v) \in S_m \times S_m \mid \{\sigma(k+1), \dots, \sigma(m)\} \subset \{v(r+k-m+1), \dots, v(m)\}\}$$

for the appropriate set of restricted permutations, (83) can equivalently be written as

$$\begin{aligned} & \frac{1}{(m-k)!(r+k-m)!(2m-r-k)!} \sum_{\sigma, v \in I_2(m, k, r)} \sum_{\tau \in S_r} (-1)^{\sigma + \tau + v} \cdot \\ & \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \langle f_{\sigma(k+1)}, x_{\tau(r+k-m+1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(r)} \rangle \cdot \\ & \cdot \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r+k-m)}, x_{\tau(r+k-m)} \rangle f_{v(r+k-m+1)} \wedge \cdots \wedge f_{v(m)}. \end{aligned} \quad (84)$$

To simplify this further, observe that for any  $(\sigma, v) \in I_2(m, k, r)$  there is a unique  $\phi \in S_{2m-r-k}$ , which 'unshuffles'  $\sigma(k+1), \dots, \sigma(m)$  in  $\{v(r+k-m+1), \dots, v(m)\}$  to the left', but leaves the order in the (ordered) set  $\{v(r+k-m+1), \dots, v(m)\} \setminus \{\sigma(k+1), \dots, \sigma(m)\}$  unchanged. In particular it acts by

$$\begin{aligned} (id_{r+k-m} \times \phi)v(\{1, \dots, m\}) = \\ \{v(1), \dots, v(r+k-m), \sigma(k+1), \dots, \sigma(m), \phi v(r+1), \dots, \phi v(m)\} \end{aligned}$$

Writing

$$J_2(m, k, r) = \{(s, u) \in S_m \times S_m \mid v(r+k-m+1) = \sigma(k+1), \dots, v(r) = \sigma(m)\}$$

this 'unshuffling' gives a surjection of sets  $\pi : I_2(m, k, r) \rightarrow J_2(m, k, r)$ , with fiber over  $\pi^{-1}(\sigma, v)$  the set of all  $(\sigma, \delta) \in J_2(m, k, r)$ , such that the  $\sigma(k+1), \dots, \sigma(m)$  are somehow inside  $\{v(r+k-m+1), \dots, v(m)\}$ .

The number of ways  $\{\sigma(k+1), \dots, \sigma(m)\}$  can be a subset of the set  $\{u(r+k-m+1), \dots, u(m)\}$ , is the same as counting the number of ways to put  $(m-k)$  labeled balls into  $2m-(r+k)$  labeled slots with exclusion, which is computed by the *falling factorial*

$$P(2m-(r+k), m-k) = (2m-(r+k)) \cdot (2m-(r+k)-1) \cdots (m-r+1) = \frac{(2m-(r+k))!}{(m-r)!}.$$

Now the point is, that all terms in (84), indexed by elements of the same fiber are in fact equal. To see that, let  $(\sigma, \delta) \in \pi^{-1}(\sigma, v)$  for some  $(\sigma, v) \in J_2(m, k, r)$ . Starting with

$$\begin{aligned} & (-1)^{\sigma + v + \tau} \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \langle f_{\sigma(k+1)}, x_{\tau(r+k-m+1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(r)} \rangle \cdot \\ & \cdot \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r+k-m)}, x_{\tau(r+k-m)} \rangle \cdot \\ & \cdot f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)} \end{aligned}$$

observe that there is a unique permutation  $\phi \in S_{2m-r-k}$  with  $(id_{r+k-m} \times \phi) \circ \delta = v$ . We can reorder the cotensor factors in the previous expression according to the inverse of  $\phi$ . Since the exterior product is antisymmetric we get

$$\begin{aligned} & (-1)^{\sigma + v + \tau + \phi^{-1}} \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \langle f_{\sigma(k+1)}, x_{\tau(r+k-m+1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(r)} \rangle \cdot \\ & \cdot \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r+k-m)}, x_{\tau(r+k-m)} \rangle \cdot \\ & \cdot f_{\phi^{-1}\sigma(k+1)} \wedge \cdots \wedge f_{\phi^{-1}\sigma(m)} \wedge f_{\phi^{-1}v(r+1)} \wedge \cdots \wedge f_{\phi^{-1}v(m)} \end{aligned}$$

which we can write as

$$\begin{aligned} & (-1)^{\sigma + (id_{r+k-m} \times \phi)^{-1}v + \tau} \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \\ & \langle f_{\sigma(k+1)}, x_{\tau(r+k-m+1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(r)} \rangle \cdot \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r+k-m)}, x_{\tau(r+k-m)} \rangle \cdot \\ & \cdot f_{\phi^{-1}\sigma(k+1)} \wedge \cdots \wedge f_{\phi^{-1}\sigma(m)} \wedge f_{\phi^{-1}v(r+1)} \wedge \cdots \wedge f_{\phi^{-1}v(m)} = \end{aligned}$$

$$(-1)^{\sigma+\delta+\tau} \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \langle f_{\sigma(k+1)}, x_{\tau(r+k-m+1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(r)} \rangle \cdot \\ \cdot \langle f_{\delta(1)}, x_{\tau(1)} \rangle \cdots \langle f_{\delta(r+k-m)}, x_{\tau(r+k-m)} \rangle f_{\delta(r+k-m+1)} \wedge \cdots \wedge f_{\delta(m)}.$$

Since we can do this for any element of the same fiber, they really describe the same term in (84).

Using this we can rewrite (84) as a sum over the index set  $J_2(m, r, k)$ . This gives

$$\frac{1}{(m-k)!(m-r)!(r+k-m)!} \sum_{\sigma, v \in J_2(m, k, r)} \sum_{\tau \in S_r} (-1)^{\sigma+v+\tau} \cdot \\ \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \langle f_{\sigma(k+1)}, x_{\tau(r+k-m+1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(r)} \rangle \cdot \\ \cdot \langle f_{v(1)}, x_{\tau(1)} \rangle \cdots \langle f_{v(r+k-m)}, x_{\tau(r+k-m)} \rangle \\ f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)}.$$

And by reordering the summations over  $\tau$  we can rewrite this further into

$$\frac{1}{(m-k)!(m-r)!(r+k-m)!} \sum_{\sigma, v \in J_2(m, k, r)} \sum_{\tau \in S_r} (-1)^{\sigma+v+\tau+(m-k)(r+k-m)} \cdot \\ \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \langle f_{\sigma(k+1)}, x_{\tau(1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(m-k)} \rangle \cdot \\ \cdot \langle f_{v(1)}, x_{\tau(m-k+1)} \rangle \cdots \langle f_{v(r+k-m)}, x_{\tau(r)} \rangle \\ f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)}. \quad (85)$$

To compare this expression with (81), we have to change to a summation over the index set  $J_1(m, r, k)$ . Therefore consider the map  $\Phi : J_1(m, r, k) \rightarrow J_2(m, r, k)$  defined by  $\Phi(\sigma, v) = (\sigma, v\kappa)$  where  $\kappa \in S_m$  is given by the permutation that maps  $(r+k-m+1, \dots, r, 1, \dots, r+k-m, r+1, \dots, m)$  to  $(1, \dots, m)$ . This permutation can be realized by  $(m-k)(r+k-m)$  transposition and since  $|J_1(m, r, k)| = |J_2(m, r, k)|$  the map  $\Phi$  is a bijection. In conclusion (81) can be written as

$$\frac{1}{(m-k)!(m-r)!(r+k-m)!} \sum_{\sigma, v \in J_2(m, k, r)} \sum_{\tau \in S_r} (-1)^{\sigma+v+\tau} \cdot \\ \cdot \langle f_{\sigma(1)}, y_1 \rangle \cdots \langle f_{\sigma(k)}, y_k \rangle \cdot \langle f_{\sigma(k+1)}, x_{\tau(1)} \rangle \cdots \langle f_{\sigma(m)}, x_{\tau(m-k)} \rangle \cdot \\ \cdot \langle f_{v(m-k+1)}, x_{\tau(m-k+1)} \rangle \cdots \langle f_{v(r)}, x_{\tau(r)} \rangle \\ f_{\sigma(k+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge f_{v(r+1)} \wedge \cdots \wedge f_{v(m)}. \quad (86)$$

which is to the right side (81), which proofs equation () for simple and therefore on arbitrary arguments.  $\square$

**Corollary B.18.** *Equation (72) is only non zero for  $|x|, |y| \leq |f|$  and  $|x|+|y| \geq |f|$ .*

## APPENDIX C. THE N-PLECTIC HOMOTOPY JACOBI EQUATION

In this section, we proof the **n-plectic homotopy Jacobi equation** in dimension  $k$ :

$$\sum_{j=1}^k \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\ \cdot \{ \{ v_{\sigma(1)}, \dots, v_{\sigma(j)} \}, v_{\sigma(j+1)}, \dots, v_{\sigma(k)} \} = 0. \quad (87)$$

Here the appearance of the  $(n-1)$ -fold shifting is due to the degree relation  $|s^{n-1}v| = |sx|$  between Poisson cotensors and associated Hamilton tensors.

We proof these equations by induction on  $k$ . Since () and (), they are satisfied for all  $k \leq 3$ , which serves as the induction base.

In a nutshell the proof is roughly as follows:

The following proposition gives the key observation, necessary for the induction to work. Loosely speaking it is the reflection of the structure equation on the level of associated Hamilton tensors.

**Proposition C.1.** *Suppose  $k \in \mathbb{N}$  and that the  $n$ -plectic Jacobi equation (87) holds for all  $j \leq (k-1)$ . Then the exterior tensor*

$$\sum_{j=1}^{k-2} \sum_{\sigma \in Sh(j, (k-1)-j)} (-1)^{\sigma+j((k-1)-j)} e(s; sx_1, \dots, sx_{k-1}) \cdot x_{\{v_{\sigma(1)}, \dots, v_{\sigma(j)}, v_{\sigma(j+1)}, \dots, v_{\sigma(k-1)}\}}$$

is an element of the kernel  $\ker(\omega)$  for all Poisson cotensors  $v_1, \dots, v_k \in V$  and associated Hamilton tensors  $x_1, \dots, x_k$ .

*Proof.* Apply the de Rham differential  $d$  to the  $n$ -plectic Jacobi equation (87) in dimension  $(k-1)$ . Since  $d^2 = 0$  we get

$$\sum_{j=1}^{k-2} \sum_{\sigma \in Sh(j, (k-1)-j)} (-1)^{\sigma+j((k-1)-j)} e(\sigma; sx_1, \dots, sx_{k-1}) \cdot d\{v_{\sigma(1)}, \dots, v_{\sigma(j)}, v_{\sigma(j+1)}, \dots, v_{\sigma(k-1)}\} = 0.$$

From () we know that if the homotopy Jacobi equation holds in dimension  $j$ , then the images of  $j$ -ary Poisson bracket are Poisson cotensors. Therefore we can use the fundamental pairing () to find Hamilton tensors associated to the nested brackets and transform the last equation into

$$\sum_{j=1}^{k-2} \sum_{\sigma \in Sh(j, (k-1)-j)} (-1)^{\sigma+j((k-1)-j)} e(\sigma; sx_1, \dots, sx_{k-1}) \cdot i_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(j)}, v_{\sigma(j+1)}, \dots, v_{\sigma(k-1)}\}}} \omega = 0.$$

□

The following theorem computes the induction step.

**Theorem C.2** (Induction Step). *Let  $k \in \mathbb{N}$  be any natural number and assume that the  $n$ -plectic Jacobi equations (87) hold for all  $j \leq (k-1)$ . Then the same equation is also satisfied for  $k$ .*

*Proof.* Consider the general  $n$ -plectic Jacobi equation (87), assume that each argument  $v_i \in V$  is homogeneous and choose appropriate Hamilton tensors  $x_i$ .

Since all  $k$ -ary brackets are  $(n-1)$ -fold shifted graded antisymmetric, we can rewrite equation (87) as a sum over arbitrary permutations:

$$\frac{1}{j!(k-j)!} \sum_{j=1}^k \sum_{\sigma \in S_k} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \{v_{\sigma(1)}, \dots, v_{\sigma(j)}, v_{\sigma(j+1)}, \dots, v_{\sigma(k)}\} = 0. \quad (88)$$

Now recall that the homotopy Poisson 1-bracket  $\{\cdot\}$  is nothing but the de Rham differential. To exploit this, rephrase (88) by separating all terms, which contain the 1-bracket:

$$d\{v_1, \dots, v_k\} \quad (89)$$

$$+ \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)} e(\sigma; sx_1, \dots, sx_k) \{dv_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}\} \quad (90)$$

$$+ \frac{1}{j!(k-j)!} \sum_{j=2}^{k-1} \sum_{\sigma \in S_k} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \{v_{\sigma(1)}, \dots, v_{\sigma(j)}, v_{\sigma(j+1)}, \dots, v_{\sigma(k)}\} = 0. \quad (91)$$

According to ( ) as well as ( ), we know that this equation is satisfied for all  $k \leq 3$  and therefore we assume  $k > 3$ . In this case the homotopy Poisson  $k$ -bracket ( ) is given by recursion, which can be used to replace the bracket in the cocycle (89) by its definition:

$$d\{v_1, \dots, v_k\} = \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) (-1)^{(k-1)} di_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)}. \quad (92)$$

To rewrite the  $k$ -ary bracket in (90), observe that its definition can be broken down into two parts. One that fixes the first argument and another one which shuffles the first argument to the  $k$ -th position. After simplification we get

$$\begin{aligned} & \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+1(k-1)} e(\sigma; v_1, \dots, v_k) \{dv_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k)}\} = \\ & \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & - \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)} e(\sigma; sx_1, \dots, sx_k) \cdot \\ & \cdot (-1)^{|x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-1)}\}}|} i_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-1)}\}}} dv_{\sigma(k)}. \quad (93) \end{aligned}$$

Now insert (92) and (93) back into (89) and (90) and apply Cartans infinitesimal homotopy formula ( ). This transform (88) into

$$\begin{aligned} & \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & + \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)} e(\sigma; sx_1, \dots, sx_k) L_{x_{\{\sigma(1), \dots, \sigma(k-1)\}}} v_{\sigma(k)} \\ & + \frac{1}{j!(k-j)!} \sum_{j=2}^{k-1} \sum_{\sigma \in S_k} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\ & \cdot \{\{v_{\sigma(1)}, \dots, v_{\sigma(j)}\}, v_{\sigma(j+1)}, \dots, v_{\sigma(k)}\} = 0. \quad (94) \end{aligned}$$

To proceed we have to consider the cases  $k = 4$  and  $k = 5$  separately, since the last terms in (94) are different in those situations. For  $k = 4$  equation (94) is equal to

$$\begin{aligned} & \frac{1}{2} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}}} v_{\sigma(4)} \\ & - \frac{1}{6} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) L_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}}} v_{\sigma(4)} \\ & + \frac{1}{4} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) \{\{v_{\sigma(1)}, v_{\sigma(2)}\}, v_{\sigma(3)}, v_{\sigma(4)}\} \\ & - \frac{1}{6} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) \{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}, v_{\sigma(4)}\} = 0. \quad (95) \end{aligned}$$

To see that (95) is correct, expand the remaining nested brackets according to their definition. After simplification this gives

$$\frac{1}{2} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}}} v_{\sigma(4)} \quad (96)$$

$$- \frac{1}{6} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) L_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}}} v_{\sigma(4)} \quad (97)$$

$$- \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) i_{[x_{\sigma(3)}, [x_{\sigma(2)}, x_{\sigma(1)}]]} v_{\sigma(4)} \quad (98)$$

$$- \frac{1}{2} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) i_{[x_{\sigma(2)}, x_{\sigma(1)}]} L_{x_{\sigma(3)}} v_{\sigma(4)} \quad (99)$$

$$+ \frac{1}{6} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) L_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}}} v_{\sigma(4)} \quad (100)$$

$$- \frac{1}{2} \sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) (-1)^{|x_{\sigma(1)}|} L_{x_{\sigma(1)}} i_{[x_{\sigma(3)}, x_{\sigma(2)}]} v_{\sigma(4)} = 0. \quad (101)$$

Now (97) cancels against (100) and using equation ( ), the terms (99) and (101) can be combined into  $\sum_{\sigma \in S_4} (-1)^\sigma e(\sigma; sx_1, \dots, sx_4) i_{[x_{\sigma(3)}, [x_{\sigma(2)}, x_{\sigma(1)}]]} v_{\sigma(4)}$ . This expression vanishes together with (98), since the Schouten-Nijenhuis brackets satisfies the strict

Jacobi equation of a Gerstenhaber algebra. The vanishing of (96) follows from theorem (), proposition () and the kernel property, since equation (87) holds for  $k = 3$ . This proves (87) for  $k = 4$ .

Now suppose  $k \geq 5$ . To proceed we separate all terms in equation (94), that contains the homotopy Poisson 2-bracket.

$$\begin{aligned} & \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & + \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)} e(\sigma; sx_1, \dots, sx_k) L_{x_{\{\sigma(1), \dots, \sigma(k-1)\}}} v_{\sigma(k)} \\ & + \frac{1}{2(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \{\{v_{\sigma(1)}, v_{\sigma(2)}\}, v_{\sigma(3)}, \dots, v_{\sigma(k)}\} \end{aligned} \quad (102)$$

$$\begin{aligned} & + \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)} e(\sigma; sx_1, \dots, sx_k) \{\{v_{\sigma(1)}, \dots, v_{\sigma(k-1)}\}, v_{\sigma(k)}\} \quad (103) \\ & + \sum_{j=3}^{k-2} \frac{1}{j!(k-j)!} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\ & \cdot \{\{v_{\sigma(1)}, \dots, v_{\sigma(j)}\}, v_{\sigma(j+1)}, \dots, v_{\sigma(k)}\} = 0. \end{aligned}$$

Now consider (102) and (103) and express the nested brackets in terms of their definitions. Since  $k \geq 5$  this leads to the following identities:

$$\begin{aligned} & \frac{1}{2(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) \{\{v_{\sigma(1)}, v_{\sigma(2)}\}, v_{\sigma(3)}, \dots, v_{\sigma(k)}\} = \\ & \frac{1}{2(k-2)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-2)} e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & - \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-2)}\}}} L_{x_{\sigma(k-1)}} v_{\sigma(k)}. \\ & \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)} e(\sigma; sx_1, \dots, sx_k) \{\{v_{\sigma(1)}, \dots, v_{\sigma(k-1)}\}, v_{\sigma(k)}\} = \\ & - \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)|x_{\sigma(1)}} e(\sigma; sx_1, \dots, sx_k) L_{x_{\sigma(1)}} i_{x_{\{v_{\sigma(2)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & - \frac{1}{(k-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)} e(\sigma; sx_1, \dots, sx_k) L_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)}. \end{aligned}$$

Now using  $[x_j, x_i] = \frac{1}{2} x_{\{v_i, v_j\}}$  and equation (), we have the following additional identity

$$\begin{aligned} & - \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-1)|x_{\sigma(1)}} e(\sigma; sx_1, \dots, sx_k) L_{x_{\sigma(1)}} i_{x_{\{v_{\sigma(2)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & - \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-2)}\}}} L_{x_{\sigma(k-1)}} v_{\sigma(k)} = \\ & \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{[x_{\sigma(k-2)}, x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-2)}\}}]} v_{\sigma(k)} = \\ & \frac{1}{2(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-2)}\}, v_{\sigma(k-1)}}} v_{\sigma(k)} \end{aligned}$$

Substituting this back into expression (102) and (103) we can further rewrite the  $n$ -plectic Jacobi equation as

$$\begin{aligned} & \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & + \frac{1}{2(k-3)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-2)} e(\sigma; sx_1, \dots, sx_k) i_{x_{\{\{v_{\sigma(1)}, v_{\sigma(2)}\}, v_{\sigma(3)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & + \frac{1}{2(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{\{v_{\sigma(1)}, \dots, v_{\sigma(k-2)}\}, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \quad (104) \\ & + \frac{1}{j!(k-j)!} \sum_{j=3}^{k-2} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\ & \cdot \{\{v_{\sigma(1)}, \dots, v_{\sigma(j)}\}, v_{\sigma(j+1)}, \dots, v_{\sigma(k)}\} = 0. \end{aligned}$$

Since this equation behaves differently for  $k = 5$  and  $k > 5$ , first assume  $k = 5$ . In this case the last row of (104) can be computed by applying the definition of the nested Poisson  $k$ -brackets. After simplification this gives

$$\begin{aligned} & \frac{1}{3!2!} \sum_{\sigma \in S_5} (-1)^{\sigma+3 \cdot 2} e(\sigma; sx_1, \dots, sx_5) \{ \{ v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)} \}, v_{\sigma(4)}, v_{\sigma(5)} \} = \\ & \frac{1}{4} \sum_{\sigma \in S_5} (-1)^{\sigma} e(\sigma; sx_1, \dots, sx_5) (-1)^{|x_{\sigma(1)}|+|x_{\sigma(2)}|} i_{[x_{\sigma(2)}, x_{\sigma(1)}]} i_{[x_{\sigma(4)}, x_{\sigma(3)}]} v_{\sigma(5)} \\ & + \frac{1}{6} \sum_{\sigma \in S_5} (-1)^{\sigma} e(\sigma; sx_1, \dots, sx_5) i_{[x_{\sigma(4)}, x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}}]} v_{\sigma(5)} \end{aligned} \quad (105)$$

Since  $(-1)^{|x_1|+|x_2|} i_{[x_2, x_1]} i_{[x_4, x_3]}(\cdot) = -(-1)^{|x_3|+|x_4|} i_{[x_4, x_3]} i_{[x_2, x_1]}(\cdot)$  the terms in the second row vanish. Using  $[x_j, x_i] = \frac{1}{2} x_{\{v_i, v_j\}}$  we can substitute (105) back into (104) to rewrite the  $n$ -plectic Jacobi expression for  $k = 5$  into

$$\begin{aligned} & - \frac{1}{3!} \sum_{\sigma \in S_5} (-1)^{\sigma+1 \cdot 3} e(\sigma; sx_1, \dots, sx_5) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, v_{\sigma(4)}\}}} v_{\sigma(5)} \\ & - \frac{1}{2 \cdot 2} \sum_{\sigma \in S_5} (-1)^{\sigma+2 \cdot 2} e(\sigma; sx_1, \dots, sx_5) i_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, v_{\sigma(4)}\}}} v_{\sigma(5)} \\ & - \frac{1}{3!} \sum_{\sigma \in S_5} (-1)^{\sigma+3 \cdot 1} e(\sigma; sx_1, \dots, sx_5) i_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}, v_{\sigma(4)}}} v_{\sigma(5)}. \end{aligned}$$

Since the  $n$ -plectic Jacobi equation is satisfied for  $k = 4$ , proposition () and the kernel property (), imply that this expression vanishes and therefore that the  $n$ -plectic Jacobi equation is satisfied for  $k = 5$ .

Now suppose  $k \geq 6$  and separate all terms that contain the homotopy Poisson 3-bracket from the last row of equation (104). This gives

$$\begin{aligned} & \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^{\sigma} e(\sigma; sx_1, \dots, sx_k) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & + \frac{1}{2(k-3)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-2)} e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\ & + \frac{1}{2(k-2)!} \sum_{\sigma \in S_k} (-1)^{\sigma} e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-2)}, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \quad (106) \\ & + \frac{1}{3!(k-3)!} \sum_{\sigma \in S_k} (-1)^{\sigma+3(k-3)} e(\sigma; sx_1, \dots, sx_k) \{ \{ v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)} \}, v_{\sigma(4)}, \dots, v_{\sigma(k)} \} \\ & + \frac{1}{2(k-2)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-2)(2)} e(\sigma; sx_1, \dots, sx_k) \{ \{ v_{\sigma(1)}, \dots, v_{\sigma(k-2)} \}, v_{\sigma(k-1)}, v_{\sigma(k)} \} \\ & + \frac{1}{j!(k-j)!} \sum_{j=4}^{k-3} \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\ & \cdot \{ \{ v_{\sigma(1)}, \dots, v_{\sigma(j)} \}, v_{\sigma(j+1)}, \dots, v_{\sigma(k)} \} = 0. \end{aligned}$$

Next consider the two rows which contain the homotopy Poisson 3-bracket and apply the definition of all involved nested brackets. After simplification this gives the following two identities:

$$\begin{aligned} & \frac{1}{3!(k-3)!} \sum_{\sigma \in S_k} (-1)^{\sigma+3(k-3)} e(\sigma; sx_1, \dots, sx_k) \cdot \\ & \cdot \{ \{ v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)} \}, v_{\sigma(4)}, \dots, v_{\sigma(k)} \} = \\ & \frac{1}{3!(k-4)!} \sum_{\sigma \in S_k} (-1)^{\sigma} e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}, v_{\sigma(4)}, \dots, v_{\sigma(k-1)}}} v_{\sigma(k)} \quad (107) \end{aligned}$$

$$\begin{aligned} & + \frac{1}{2!(k-3)!} \sum_{\sigma \in S_k} (-1)^{\sigma+3(k-3)} e(\sigma; sx_1, \dots, sx_k) \cdot \\ & \cdot (-1)^{\sum_{i=1}^{k-3} |x_{\sigma(i)}|} i_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-3)}\}} i_{[x_{\sigma(k-1)}, x_{\sigma(k-2)}]} v_{\sigma(k)} \quad (108) \end{aligned}$$

$$\begin{aligned} & \frac{1}{2(k-2)!} \sum_{\sigma \in S_k} (-1)^{\sigma} e(\sigma; sx_1, \dots, sx_k) \{ \{ v_{\sigma(1)}, \dots, v_{\sigma(k-2)} \}, v_{\sigma(k-1)}, v_{\sigma(k)} \} = \\ & + \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^{\sigma} e(\sigma; sx_1, \dots, sx_k) i_{[x_{\sigma(k-1)}, x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-2)}\}}]} v_{\sigma(k)} \end{aligned}$$



$$\begin{aligned}
& + \frac{1}{(k-3)!2!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-3)} e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot (-1)^{(k-2)(|x_{\sigma(1)}|+|x_{\sigma(2)}|)} i_{[x_{\sigma(2)}, x_{\sigma(1)}]} i_{x_{\{v_{\sigma(3)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \quad (109)
\end{aligned}$$

From  $i_x i_{x'} = (-1)^{|x||x'|} i_{x'} i_x$  follows that (108) cancels against (109). Insert the remaining terms back into (106) and use  $[x_j, x_i] = \frac{1}{2} x_{\{v_i, v_j\}}$  gives the following expression for the  $n$ -plectic Jacobi identity:

$$\begin{aligned}
& \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{dv_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\
& + \frac{1}{2(k-3)!} \sum_{\sigma \in S_k} (-1)^{\sigma+k} e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}\}, v_{\sigma(3)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\
& + \frac{1}{3!(k-4)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}\}, v_{\sigma(4)}, \dots, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \\
& + \frac{1}{(k-2)!} \sum_{\sigma \in S_k} (-1)^\sigma e(\sigma; sx_1, \dots, sx_k) i_{x_{\{v_{\sigma(1)}, \dots, v_{\sigma(k-2)}\}, v_{\sigma(k-1)}\}}} v_{\sigma(k)} \quad (110) \\
& + \frac{1}{j!(k-j)!} \sum_{j=4}^{k-3} \sum_{\sigma \in S_k} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot \{v_{\sigma(1)}, \dots, v_{\sigma(j)}\}, v_{\sigma(j+1)}, \dots, v_{\sigma(k)}\} = 0.
\end{aligned}$$

To rewrite the last row in (110), suppose  $4 \leq j \leq k-3$  is fixed and rephrase the outer Poisson bracket in terms of its definition.

$$\begin{aligned}
& \sum_{\sigma \in Sh(j, k-j)} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot \{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\} = \\
& \frac{1}{j!(k-j-1)!} \sum_{\sigma \in S_k} (-1)^{\sigma+(j+1)(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k-1)}\}}} f_{\sigma(k)} \\
& + \frac{1}{j!(k-j)!} \sum_{\sigma \in S_k} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot (-1)^j \sum_{i=1}^{k-j} |x_{\sigma(i)}| i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k-j)}\}}} \{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}\} \cdot \quad (111)
\end{aligned}$$

Expanding the remaining Poisson bracket in (111), using  $i_x i_{x'} = (-1)^{|x||x'|} i_{x'} i_x$  leads to the following identity

$$\begin{aligned}
& + \frac{1}{j!(k-j)!} \sum_{\sigma \in S_k} (-1)^{\sigma+j(k-j)} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^j \sum_{i=1}^{k-j} |x_{\sigma(i)}| i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(k-j)}\}}} \{f_{\sigma(k-j+1)}, \dots, f_{\sigma(k)}\} = \\
& - \frac{1}{(k-j)!j!} \sum_{\sigma \in S_k} (-1)^{\sigma+(k-j)j} e(\sigma; sx_1, \dots, sx_k) \\
& \quad (-1)^{(k-j) \sum_{i=1}^j |x_{\sigma(i)}|} i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}}} \{f_{\sigma(j+1)}, \dots, f_{\sigma(k)}\}
\end{aligned}$$

and it follows that for given  $j$  each term (111) cancel against the same term for  $(k-j)$ . Taking this into account the left side of the  $n$ -plectic Jacobi equation (88) can finally be written as

$$\begin{aligned}
& \sum_{j=1}^{k-1} \frac{1}{j!((k-1)-j)!} \sum_{\sigma \in S_k} (-1)^{\sigma+j((k-1)-j)} e(\sigma; sx_1, \dots, sx_k) \cdot \\
& \quad \cdot i_{x_{\{f_{\sigma(1)}, \dots, f_{\sigma(j)}\}, f_{\sigma(j+1)}, \dots, f_{\sigma(k-1)}\}}} v_{\sigma(k)}
\end{aligned}$$

By the induction hypothesis, equation (88) is satisfied for  $(k-1)$  and proposition () as well as the kernel property () imply that the previous expression vanishes and therefore that equation (88) is satisfied for  $k$ , too. This completes the induction.  $\square$

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